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Complexity Theory

Classify problems according to the **computational resources** required

- running time
- storage space
- parallelism
- randomness
- rounds of interaction, communication, others…

Attempt to answer: what is computationally feasible with limited resources?

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Complexity Theory

- Contrast with decidability: What is computable?
  - answer: some things are not
- We care about resources!
  - leads to many more subtle questions
  - fundamental open problems

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The central questions

- Is finding a solution as easy as recognizing one?
  P = NP?
- Is *every* efficient sequential algorithm parallelizable?
  P = NC?
- Can *every* efficient algorithm be converted into one that uses a tiny amount of memory?
  P = L?
- Are there small Boolean circuits for *all* problems that *require* exponential running time?
  EXP ⊆ P/poly?
- Can *every* efficient randomized algorithm be converted into a deterministic algorithm one?
  P = BPP?

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Central Questions

We *think* we know the answers to all of these questions …

… **but** no one has been able to prove that even a small part of this "world-view" is correct.

If we're wrong on any one of these then computer science will change dramatically

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Introduction

- You already know about two complexity classes
  - **P** = the set of problems decidable in *polynomial time*
  - **NP** = the set of problems with witnesses that can be checked in polynomial time

  … and notion of *NP-completeness*
- Useful **tool**
- Deep **mathematical problem**: P = NP?

Course should be **both** useful **and** mathematically interesting

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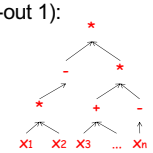
A question

- Given: polynomial $f(x_1, x_2, \ldots, x_n)$ as arithmetic formula (fan-out 1):
  - multiplication (fan-in 2)
  - addition (fan-in 2)
  - negation (fan-in 1)



- **Question**: is f identically zero?

(variables take values in finite field of size > degree)

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A question

- **Given**: multivariate polynomial $f(x_1, x_2, \ldots, x_n)$ as an arithmetic formula.
- **Question**: is f identically zero?
- Challenge: devise a deterministic poly-time algorithm for this problem.

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A randomized algorithm

- **Given**: multivariate degree r poly. $f(x_1, x_2, \ldots, x_d)$
  note: $r = \deg(f) \leq$ size of formula
- **Algorithm**:
  - pick small number of random points
  - if f is zero on all of these points, answer "yes"
  - otherwise answer "no"

(low-degree non-zero polynomial evaluates to zero on only a small fraction of its domain)

- No efficient deterministic algorithm known

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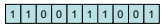
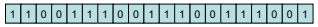
Derandomization

- Here is a deterministic algorithm that works under the assumption that there exist hard problems, say SAT.
- solve SAT on all instances of length log n



- encode using error-correcting code (variant of a Reed-Muller code)



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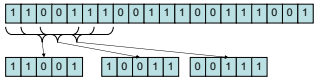
Derandomization



- run randomized alg. using these strings in place of random evaluation points
  - if f is zero on all of these points, answer "yes"
  - otherwise answer "no"
- This works. (proof in this course)

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Derandomization

This technique works on any randomized algorithm.

Gives generic "derandomization" of randomized procedures.

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A surprising fact

- Is finding a solution as easy as recognizing one?
  P = NP? probably FALSE
- Is *every* sequential algorithm parallelizable?
  P = NC? probably FALSE
- Can *every* efficient algorithm be converted into one that uses a tiny amount of memory?
  P = L? probably FALSE
- Are there small Boolean circuits for *all* problems that *require* exponential running time?
  EXP ⊆ P/poly? probably FALSE
- Can *every* randomized algorithm be converted into a deterministic algorithm one?
  P = BPP? **probably TRUE**

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Outline

Should be mostly review…

1. Problems and Languages
2. Complexity Classes
3. Turing Machines
4. Reductions
5. Completeness

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Problems and Languages

- Need formal notion of "computational problem". Examples:
  - Given graph G, vertices s, t, find the shortest path from s to t
  - Given matrices A and B, compute AB
  - Given an integer, find its prime factors
  - Given a Boolean formula, find a satisfying assignment

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Problems and Languages

- One possibility: function from strings to strings

  $f:\Sigma^* \to \Sigma^*$
- **function problem**:
  given x, compute f(x)
- **decision problem**: $f:\Sigma^* \to \{yes, no\}$
  given x, accept or reject

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Problems and Languages

- simplification doesn't give up much:
  - Given an integer n, find its prime factors
  - Given an integer n and an integer k, is there a factor of n that is < k?
  - Given a Boolean formula, find a satisfying assignment
  - Given a Boolean formula, is it satisfiable?
- can solve function problem efficiently using related decision problem (how?)
- **We will work mostly with decision problems**

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Problems and Languages

- **decision problems**: $f:\Sigma^* \to \{yes, no\}$
- equivalent notion: **language** $L \subseteq \Sigma^*$
  L = set of "yes" instances
- Examples:
  - set of strings encoding satisfiable formulas
  - set of strings that encode pairs (n,k) for which n has factor < k
- decision problem associated with L:
  - Given x, is x in L?

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Problems and Languages

An aside: two encoding issues

1. implicitly assume we've agreed on a way to encode inputs (and outputs) as strings
   - sometimes relevant in fine-grained analysis (e.g. adj. matrix vs. adj. list for graphs)
   - almost never an issue in this class
   - avoid silly encodings: e.g. unary

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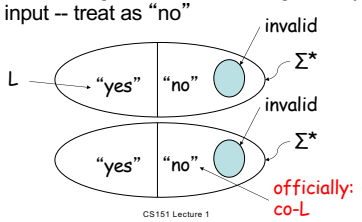
Problems and Languages

2. some strings not valid encodings of any input -- treat as "no"



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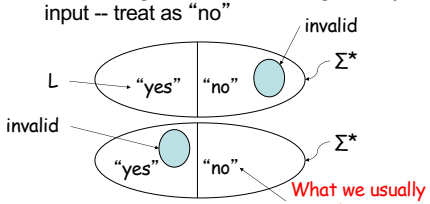
Problems and Languages

2. some strings not valid encodings of any input -- treat as "no"



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Complexity Classes

- **complexity class** = class of languages
- set-theoretic definition – no reference to computation (!)
- example:
  - **TALLY** = languages in which every yes instance has form 0^n
  - e.g. $L = \{ 0^n : n \text{ prime} \}$

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Complexity Classes

- complexity classes you know:
  - **P** = the set of languages decidable in *polynomial time*
  - **NP** = the set of languages L where

    $L = \{ x : \exists\, y, |y| \le |x|^k, (x, y) \in R \}$

    and R is a language in **P**
- easy to define complexity classes…

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Complexity Classes

- …harder to define **meaningful** complexity classes:
  - **capture** genuine computational phenomenon (e.g. parallelism)
  - **contain** natural and relevant problems
  - ideally **characterized** by natural problems (completeness – more soon)
  - **robust** under variations in model of computation
  - possibly **closed** under operations such as AND, OR, COMPLEMENT…

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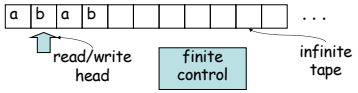
Complexity Classes

- need a **model of computation** to define classes that capture important aspects of computation
- Our model of computation: Turing Machine



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Turing Machines

- **Q** finite set of states
- ∑ alphabet including blank: "_"
- q_{start}, q_{accept}, q_{reject} in Q
- **δ** : Q x ∑ → Q x ∑ x {L, R, -} transition fn.
- input written on tape, head on 1st square, state q_{start}
- sequence of steps specified by **δ**
- if reach q_{accept} or q_{reject} then halt

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Turing Machines

- three notions of computation with Turing machines. In all, input x written on tape…
  - **function computation**: output f(x) is left on the tape when TM halts
  - **language decision**: TM halts in state q_{accept} if x ∈ L; TM halts in state q_{reject} if x ∉ L.
  - **language recognition**: TM halts in state q_{accept} if x ∈ L; may loop forever otherwise.

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Example:

q	σ	δ(q,σ)
start	0	(start, 0, R)
start	1	(start, 1, R)
start	_	(t, _, L)
start	#	(start, #, R)

q	σ	δ(q,σ)
t	0	(accept, 1, -)
t	1	(t, 0, L)
t	#	(accept, #, R)

#	0	1								start
#	0	1								start
#	0	1								start
#	0	1								start
#	0	1								t
#	0	0								t
#	1	0								accept

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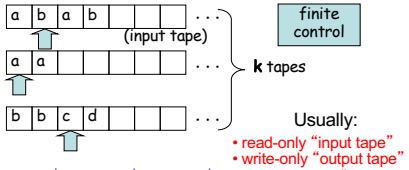
Turing Machines

- **multi-tape** Turing Machine:



Usually:
- read-only "input tape"
- write-only "output tape"
- k-2 read/write "work tapes"

$\delta: Q \times \Sigma^k \rightarrow Q \times \Sigma^k \times \{L,R,-\}^k$

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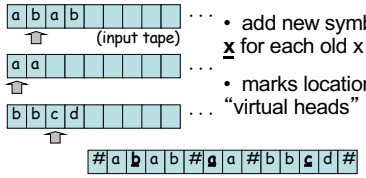
Multitape TMs

simulation of k-tape TM by single-tape TM:



- add new symbol **x** for each old x
- marks location of "virtual heads"

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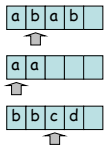
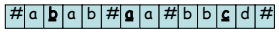
Multitape TMs



Repeat: O(t(n)) times

- scan tape, remembering the symbols under each virtual head in the state
  O(k t(n)) = O(t(n))
- make changes to reflect 1 step of M; if hit #, shift to right to make room.
  O(k t(n)) = O(t(n))

when M halts, erase all but output string
O(k t(n)) = O(t(n))



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Extended Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an efficient algorithm is:

The "extended" Church-Turing Thesis

everything we can compute in time t(n) on a physical computer can be computed on a Turing Machine in time $t^{O(1)}(n)$ (polynomial slowdown)

- quantum computers challenge this belief

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Extended Church-Turing Thesis

- consequence of extended Church-Turing Thesis: all reasonable physically realizable models of computation can be *efficiently* simulated by a TM
- e.g. multi-tape vs. single tape TM
- e.g. RAM model

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Turing Machines

- Amazing fact: there exist (natural) **undecidable** problems

HALT = { (M, x) : M halts on input x }

- Theorem: HALT is undecidable.

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Turing Machines

- Proof:
  - Suppose TM H decides HALT
  - Define new TM H' : on input <M>
    - if H accepts (M, <M>) then loop
    - if H rejects (M, <M>) then halt
  - Consider H' on input <H'>:
    - if it halts, then H rejects (H' , <H' >), which implies it cannot halt
    - if it loops, then H accepts (H' , <H' >) which implies it must halt
  - contradiction.

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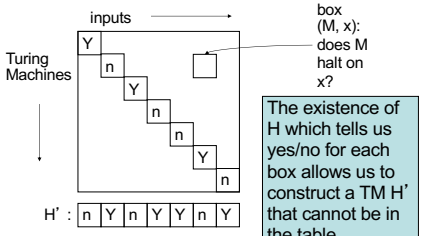
Diagonalization



The existence of H which tells us yes/no for each box allows us to construct a TM H' that cannot be in the table.

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