ACM 201 Problem Set 4

February 26, 2011

1. Please write down your solutions clearly and concisely, bind pages in order.

2. Submit your copy in class or drop it in Mulin’s mail box located at Firestone lobby by Tuesday, 6:00pm, Mar 1st, 2011.

3. Extensions are ONLY granted in medical or extreme cases. Please contact directly TA at least one day before the due day for extension.

4. Should you have questions regarding the homework, please contact TA directly. Mulin can be reached
   (a) by sending emails to mulinch@caltech.edu with subject line containing ACM201 Questions.
   (b) by attending Office Hour at SFL 2-1 from 3:00pm to 4:00pm every Monday
   (c) by calling x4537
   (d) by stopping by his office at Fireston 216

   The first two methods are preferred and the last two methods are less preferred, but still are welcomed. Responses to last-minute questions can not be guaranteed.

5. Honor Code is in effect. You may talk to, or collaborate with each other, but you must write solutions by yourself.

6. Problems are not placed in order of difficulty, so start with ones you feel comfortable first and come back to the sticky ones.

7. Good Luck.
Problem 1

Complete Integral of Hamilton-Jacobi Equation

[Problem 1, Page 162, textbook by Evans]

Prove

\[ u(x, t, a, b) = a \cdot x - tH(a) + b \quad (a \in \mathbb{R}^n, b \in \mathbb{R}) \tag{1} \]

is a complete integral of the Hamilton-Jacobi equation

\[ u_t + H(Du) = 0 \tag{2} \]

Problem 2

Another Way to Derive formula (5)

[Problem 2, Page 162, textbook by Evans]

1. Write down the characteristic equations for the PDE

\[ u_t + b \cdot Du = f \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \tag{3} \]

where \( b \in \mathbb{R}^n, f = f(x, t) \).

2. Use the characteristic ODE to solve (3) subject to the initial condition

\[ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\}. \]

Make sure your answer agrees with formula (5) in section 2.1.2

\[ u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s) \, ds \quad (x \in \mathbb{R}^n, t \geq 0) \tag{4} \]

Problem 3

Examples of Characteristics

[Problem 3, Page 163, textbook by Evans]

Solve using characteristics:

1. \[
\begin{aligned}
x_1 u x_1 + x_2 u x_2 &= 2u \\
u(x_1, 1) &= g(x_1)
\end{aligned}
\] \tag{5} 

2. \[
\begin{aligned}
u u x_1 + u x_2 &= 1 \\
u(x_1, x_1) &= \frac{1}{2} x_1
\end{aligned}
\] \tag{6} 

3. \[
\begin{aligned}
x_1 u x_1 + 2x_2 u x_2 + u x_3 &= 3u \\
u(x_1, x_2, 0) &= g(x_1, x_2)
\end{aligned}
\] \tag{7}
**Problem 4**

**Scalar Conservation Law**

*Problem 4, Page 163, textbook by Evans*

Verify that formula (61) in section 3.2.5 provides an implicit solution of the scalar conservation law of $u$,

\[
u = g(x - tF'(u)).
\]

\[(8)\]

\[\\left\{
\begin{array}{ll}
u_t + F'(u) \cdot Du = 0 & \text{in } U = \mathbb{R}^n \times (0, \infty) \\
u = g & \text{on } \Gamma = \mathbb{R}^n \times \{t = 0\}
\end{array}
\right.
\]

**Problem 5**

**Non-Characteristic Condition For General Quasilinear PDE System**

The general $m$th-order quasilinear PDE system for a vector-valued function $u(x) = (u_1(x), u_2(x), \cdots, u_N(x))^T : \mathbb{R}^n \to \mathbb{R}^N$, can be written as

\[
L(u) = \sum_{|\alpha|=m} A_\alpha D^\alpha u + C = 0
\]

\[(10)\]

where the derivative $D^\alpha u = (D^\alpha u_1, D^\alpha u_2, \cdots, D^\alpha u_N)^T$ is understood as a column vector and $A_\alpha$ and $C$ are respectively $N \times N$ matrix-valued and $N \times 1$ vector-valued functions of independent variables $x_i$, $i = 1, 2, \cdots, n$, and $p^\beta_j = D^\beta u_j$, $|\beta| \leq m - 1$, $j = 1, 2, \cdots, N$. A hyper-surface $S \subset \mathbb{R}^n$ is given by

\[
\phi(x_1, x_2, \cdots, x_n) = 0
\]

\[(11)\]

where we assume $\phi$ has $m$th continuous derivatives and

\[|D\phi| \neq 0 \quad \forall x \in S\]

In this problem, we are concerned with the solution of \ref{22} which has prescribed Cauchy data on the hyper-surface $S$. Here the Cauchy data refers to the derivatives of $u$ of orders less than $m$ and such problem is called the Cauchy problem.

1. Show that the number of different $k$th order derivatives of $u$ is

\[
\# \{|\beta| = k\} = \frac{(n + k - 1)!}{(n-1)!k!}
\]

\[(12)\]

2. The general nonlinear PDE system

\[
F(D^\alpha u, x) = 0
\]

\[(13)\]

can be reduced formally, assuming $F$ is smooth, to quasilinear ones by applying a first-order differential operator.

3. The Cauchy data can not be assigned arbitrarily, but have to satisfy some compatibility conditions on the hyper-surface $S$, which is the main purpose of this problem. We call the hyper-surface $S$ is non-characteristic at a point $x \in S$ if all $D^\alpha u(x)$ for $|\alpha| = m$ on $S$ can be found uniquely from the Cauchy data by solving \ref{22}. $S$ is call characteristic if it is not non-characteristic at all $x \in S$. We want to derive some algebraic conditions for the characteristic surface, starting by considering a simple case: the hyper-surface $S$ is the coordinate plan, i.e., $x_n = 0$, without loss of generality. Show that the desired non-characteristic condition at $x \in S$ is

\[
\det(A_{\alpha^*}) \neq 0
\]

\[(14)\]

where $\alpha^* = (0, 0, \cdots, m)$. Furthermore, explain the different implications of this non-characteristic condition for the linear and quasilinear PDE systems, respectively.
4. Note that the above condition depends on which coordinate plane the hyper-surface $S$ is aligned to. To extend the condition (26) for more general hyper-surfaces, we start by finding some intrinsic form of the non-characteristic condition (26). To this end, we define

\[ \Lambda(\xi) = \sum_{|\alpha| = m} A_{\alpha} \xi^\alpha \]  
\[ Q(\xi) = \det(\Lambda(\xi)) \]

where $\xi \in \mathbb{R}^n$, $\xi^\alpha = \prod_{i=1}^n \xi_i^{\alpha_i}$ and $Q(\xi)$ is also called the characteristic form. To simplify notations, we have intentionally omitted their dependence on independent variables $x_i$, $i = 1, 2, \ldots, n$, and $p^\alpha_j = D^\beta u_j$, $|\beta| \leq m - 1$, $j = 1, 2, \ldots, N$. Show that the non-characteristic condition (26) can be written in an intrinsic form, here we still assume $S$ is the coordinate plane.

\[ Q(D\phi) \neq 0 \]  

5. For the Cauchy data defined on the general hyper-surface $S$, i.e., being not aligned to any coordinate planes, the non-characteristic condition can again be written as (29).