Lecture 10 - A test for n qubits

Tuesday, December 1, 2020 9:16 AM

Last time: "reduced" $MIP^* = RE$ to:

<u>Claim</u>: $\forall n \geq 1$, there is a nonlocal game G_n st "G_n is a test for n qubits".

<u>Desiderata</u>

(i) Success in the game implies large dimension $(\geq (1-O(\varepsilon))\cdot 2^n)$

(ii) G_n should be "efficient" $\#Q, \#A \sim \text{poly}(n)$.

(iii) testing property should be "robust"
we want consequences from any strategy that succeeds wp $\geq 1-\varepsilon$.

A - n qubits.

<u>Def</u> $n \geq 1$ n qubits are specified by $2n$ observables $X_1, Z_1, \ldots, X_n, Z_n$ on H st

$\forall i \quad \{X_i, Z_i\} = 0$

$\forall i \neq j \quad [X_i, Z_j] = [X_i, X_j] = \cdots = 0.$

<u>Claim</u> n qubits are isomorphic to $\sigma_{X,1}, \sigma_{Z,1}, \ldots, \sigma_{X,n}, \sigma_{Z,n}$

$H \simeq (\mathbb{C}^2)^{\otimes n} \otimes H'$.

Pauli on ith copy of $\mathbb{C}^2$

<u>Def</u> $n=1$ $\varepsilon \geq 0$

An ε-approx qubit is $(|\psi\rangle, X, Z)$

$|\psi\rangle \in H$ X, Z obs on H $\| \{X, Z\}|\psi\rangle \| \leq \varepsilon$

Claim ε-approx qubit $\Rightarrow$ $\exists V: H \to \mathbb{C}^2 \otimes H'$ isometry

$$\| V X |\psi\rangle - \sigma_x \otimes \mathbb{1}\, V|\psi\rangle \| \leq O(\sqrt{\varepsilon})$$

Def n "state-dep't" qubit are $|\psi\rangle$ X_i, Z_i

st $\forall i$ $\{X_i, Z_i\}|\psi\rangle = 0$

$\forall i \neq j$ $[X_i, X_j]|\psi\rangle = [X_i, Z_j]|\psi\rangle = \cdots = 0.$

Obs $\exists$ n state-dpd qubit st $\dim(H) = O(n^2)$

$|\psi\rangle = |0 \cdots 0\rangle \in (\mathbb{C}^2)^{\otimes n}$

$\tilde{X}_i |\psi\rangle = |0 \cdots 1\, 0 \cdots 0\rangle$
$\uparrow_i$

$H = \mathrm{span}\{ |0 \cdots 0\rangle, |0 \cdots 0\, 1\, 0 \cdots 0\rangle, |0 \cdots 0\, 1\, 0 \cdots 0\, 1\, 0 \cdots 0\rangle \}.$

$\tilde{X}_k \tilde{X}_j \tilde{X}_i |\psi\rangle = |0 \cdots 1\, 0 \cdots 1\, 0 \cdots 0\rangle$
$j \uparrow$ $\uparrow_i$

Def n ε-approx qubits are observables X_i, Z_i

st $\|\{X_i, Z_i\}\| \leq \varepsilon$ $\forall i$

$\|[X_i, Z_j]\|, \ldots, \leq \varepsilon.$ $\forall i \neq j$

$\uparrow$ operator norm.

Claim $\forall \varepsilon$, $\exists$ n ε-approximate qubits on H

st $\dim(H) = \min\{ \underline{n^{O(1/\varepsilon^2)}}, 2^{(1-\varepsilon)n} \}.$

n-qubit Pauli group P_n.

Def P_n is the group generated by $\sigma_{x,1}, \sigma_{z,1}, \ldots, \sigma_{x,n}, \sigma_{z,n}$ on $(\mathbb{C}^2)^{\otimes n}$.

P_n has $2 \cdot 4^n$ elements

$P_n = \{ \pm \sigma_x(a) \sigma_z(b) : a, b \in \{0,1\}^n \}$

$\sigma_x(a) \to \sigma_{x,1}^{a_1} \otimes \cdots \otimes \sigma_{x,n}^{a_n}$

$\sigma_z(b) \to \sigma_z^{b_1} \otimes \cdots \otimes \sigma_z^{b_n}$

$\sigma_{x,1} \otimes \dots \otimes \sigma_{x,n}$

$\hookrightarrow \sigma_{z,1}^{b_1} \otimes \dots \otimes \sigma_{z,n}^{b_n}$

ex : $n = 1$. $|P_1| = 8 = \{ \mathbb{1}, -\mathbb{1}, \sigma_x, -\sigma_x, \sigma_z, -\sigma_z, \sigma_x\sigma_z, -\sigma_x\sigma_z \}$.

Def : $n \geq 1$ $\varepsilon \geq 0$. n ε-approx qubits are given by $|\psi\rangle \in H$ and observables $\{ Q(a,b) : a,b \in \{0,1\}^n \}$ on H s.t

$\forall (a,b)$ $\forall (a',b')$ $\| (\underbrace{Q(a,b)\, Q(a',b')}_{\sigma_x(a)\sigma_z(b)\sigma_x(a')\sigma_z(b')} - (-1)^{a'\cdot b} Q(a+a', b+b')) |\psi\rangle \| \leq \varepsilon$

Rk if we define $X_i = Q(e_i, 0)$
$Z_i = Q(0, e_i)$

then $\| (X_i Z_i + Z_i X_i) |\psi\rangle \| \leq \varepsilon$

$$\boxed{\sigma_x(a) \cdot \sigma_z(b) = (-1)^{a \cdot b} \sigma_z(b) \sigma_x(a)}$$

B- Approximate group representations.

Finite group G.

Def : $f : G \to U_d(\mathbb{C})$ ← unitaries on $\mathbb{C}^d$

is a representation if

$\forall x, y \in G$ $f(xy) = f(x) f(y)$.

$\Rightarrow f(x^{-1}) = f(x)^{-1} = f(x)^*$

$\Leftrightarrow f(x^{-1}y) = f(x)^* f(y)$

ex : $G = P_1$ exact qubit $\underline{X, Z}$: obs. on H st $\{X, Z\} = 0$

Define $f : P_1 \to U(H)$ ← unitaries on H

$\pm \mathbb{1} \mapsto \pm \mathbb{I}$

$\pm \sigma_x \mapsto \pm X$

$\pm \sigma_z \mapsto \pm Z$

$\pm \sigma_x\sigma_z \mapsto \pm XZ$

Then f is a representation of P_1.

Then f is a representation of P_1.

Def $\varepsilon \geq 0$, $\sigma \in \text{Density}(H)$ $\sigma \geq 0$ $\text{Tr}(\sigma) = 1$.

An (ε, σ)-rep of G is

$$f : G \to U(H)$$

s.t.
$$\mathop{E}_{x,y \in G} \| f(x)^* f(y) - f(x^{-1}y) \|_\sigma^2 \leq \varepsilon$$

uniform average

$\|A\|_\sigma^2 = \text{Tr}(A^* A \sigma)$.

$\| \{X, Z\} |\psi\rangle \|^2 \leq \varepsilon^2$

$= \text{Tr}(\{X,Z\}^2 \cdot |\psi\rangle\langle\psi|) = \| \{X,Z\} \|^2_{|\psi\rangle\langle\psi|}$

(Rk if $\sigma = \frac{1}{d} \mathbb{I}$
then $\|A\|_\sigma^2 = \frac{1}{d} \|A\|_F^2$)

Rk: Our definition of n ε-approximate qubits immediately gives an $(O(\varepsilon^2), \sigma)$-rep of P_n where $\sigma = |\psi\rangle\langle\psi|$.

Th (Gowers-Hatami '18) case $\sigma = \frac{1}{d}\mathbb{I}$.

G finite gp. f an (ε, σ)-representation of G on $\mathbb{C}^d$

Then $\exists\, d'$ st $\underline{d \leq d' \leq (1 + \underbrace{O(\varepsilon)}_{<1})\, d}$

and $V : \mathbb{C}^d \to \mathbb{C}^{d'}$

and $g : G \to U(\mathbb{C}^{d'})$ an exact rep. of G

st
$$\mathop{E}_{x \in G} \| f(x) - V^* g(x) V \|_\sigma^2 = O(\varepsilon)$$

"approx reps are close to exact representations"

Rk: • Generalizes BLR linearity test (case $G = \mathbb{Z}_2^n$)

$f : \{0,1\}^n \to \{\pm 1\}^d$ $(f_1, \dots, f_d)$

ε-approx $\Rightarrow$ on avg i f_i is ε-approx rep

$G = P_n$ what are exact representations of G?

- one representation in dim 2^n : usual matrix representation.
- 4^n representations in dim 1.

$$f(-1) = 1$$

and $\left.\begin{aligned} f(\sigma_{x,i}) &= \pm 1 \\ f(\sigma_{z,i}) &= \pm 1 \end{aligned}\right\} 4^n$ choices.

<u>Corollary of thm</u>. If $G = P_n$ and $\{ Q(a,b) : \{a,b\} \in \{0,1\}^n \}$ are observables on $\mathbb{C}^d$ s.t they form n ε-approx qubits.

Then (Th) $\Rightarrow$ $\left[\begin{aligned} &\exists \quad d \le d' \le (1+O(\varepsilon))\, d \\ &V : \mathbb{C}^d \to \mathbb{C}^{d'} \end{aligned}\right.$ $\qquad d' = 2^n \cdot k$.

st
$$\mathop{E}_{a,b \in \{0,1\}^n} \| Q(a,b) - V^\dagger \sigma_x(a)\sigma_z(b) V \|^2_{|\psi\rangle\langle\psi|} \le O(\varepsilon)$$

$g =$