Lecture 8: Tensor networks and the detectability lemma

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In this class, we will discuss the *detectability lemma*, which can be seen as a step towards constructing a gap amplification procedure for local Hamiltonians. Before this, we describe the tensor network formalism and its applications to solving some "easy" cases of the gapped local Hamiltonian problem. The tensor network formalism will be useful in future discussions in the class.

8.1 Tensor networks

Tensor networks provide a graphical method for manipulating tensors. A tensor network for a rank k tensor u living in $\bigotimes_{i=1}^{k} \mathbb{C}^{d_i}$ is represented by a dot with k *wires* attached to the dot. Assigning integral values c_i (where $1 \leq c_i \leq d_i$) to the wires yields the entry $u_{c_1,c_2,\ldots,c_k}$. Tensor product of two tensors is represented simply by drawing the two tensors side-by-side: the interpretation being that the output in this case is the product of the two "dots" corresponding to the two tensors. The more interesting operation is *contraction*, in which two wires belonging to different (or same) tensors are "fused" together. The interpretation for this is that the new output for a fixed setting of the other wires is the sum of all outputs over all possible setting of equal values to the two fused wires (the two fused wires must therefore have the same dimension).

The above rules allow common operations such as matrix multiplication to be represented succinctly, as shown in fig. 1. The labels on the wires represent the dimensions.

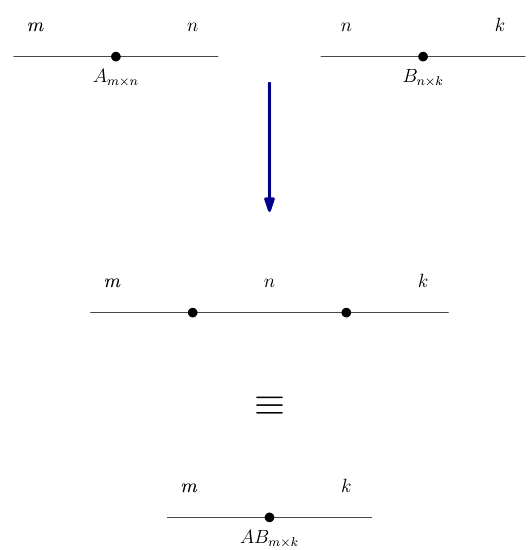

Figure 1: Tensor network representation of matrix multiplication

For our purposes, the most relevant tensor network representation is for the *Schmidt decomposition*. Recall that given a vector $|u\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, the Schmidt decomposition procedure guarantees a representation of the form

$$|u\rangle = \sum_{k=1}^{\min(d_1,d_2)} \lambda_k |u_k\rangle |v_k\rangle,$$

where the $|u_k\rangle$ and $|v_k\rangle$ are sets of orthonormal vectors (i.e., $\langle u_i | u_j\rangle = \langle v_i | v_j\rangle = \delta_{ij}$) and the λ_k are *positive* real numbers. Up to an adjoint (or the identification of a finite dimensional vector space with its dual), the Schmidt decomposition can also be written as

$$|u\rangle = \sum_{k=1}^{\min(d_1,d_2)} \lambda_k |u_k\rangle \langle v_k|,$$

which has the tensor network given in fig. 2. Note that the matrix $\sum_k \lambda_k |u_k\rangle\langle v_k|$ is diagonal (in

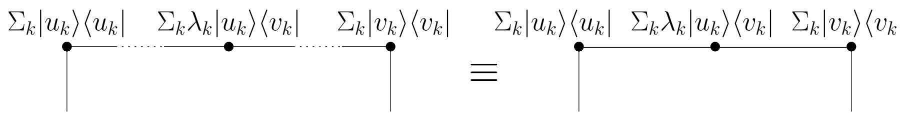

Figure 2: Schmidt decomposition

the appropriate bases), and the dimensions of the contracted tensors are at most $\min(d_1, d_2)$. This therefore gives a more compact representation of the state, if, e.g., $d_2 \gg d_1$. To take advantage of this, we can try to use the Schmidt decomposition iteratively for multi-partite states as well. This is described in fig. 3.

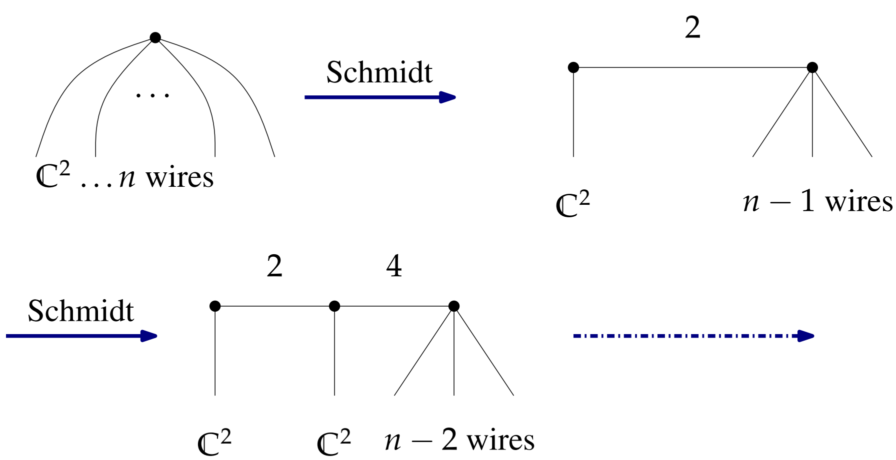

Figure 3: Iterative Schmidt decomposition

The maximum dimension obtained on one of the contracted wires in the top row at the end of such a iterative decomposition is known as the *bond dimension* of the decomposition. In general, if the bond dimension is B, this gives a representation of the state in terms of $O(nB^2)$ parameters.

However, repeating the above procedure for an arbitrary state can produce bond dimensions which are $2^{n/2}$. This is tight, since an exact description of a state in $\mathbb{C}^{2^n}$ indeed requires these many parameters. Nevertheless, there are interesting classes of states which admit such decompositions with a small bond dimension. A decomposition with a small bond dimension allows interesting properties of the state (such as its energy with respect to a local Hamiltonian) to be computed using relatively simple classical algorithms (see below), and hence characterizing states which have such decomposition is of great importance. For example, if the ground state of a given local Hamiltonian admits such a decomposition, then the above discussion implies that its ground state is easy to compute (classically), and hence we cannot expect the ground states of such Hamiltonians to be QMA-hard to compute.

8.1 Computing the energy of a state with small bond dimension

As a simple example, let us see how to compute the energy of a state $|\psi\rangle$ with respect to a Hamiltonian H acting on 2 adjacent qubits. Let us assume that $|\psi\rangle$ has a tensor network representation with bond dimension B. Now, we consider the tensor network representation of the energy $\langle\psi|H|\psi\rangle$, given in fig. 4. Consider the first blue line on the left. The tensor to its left is B^2 dimensional, can be computed in time $O(B^3)$ via two matrix multiplications. Once this is done, we can inductively compute the tensor to the left of the *second* blue line, again in $O(B^3)$ operations. Working in this way, the whole network can be evaluated in time $O(nB^3)$.

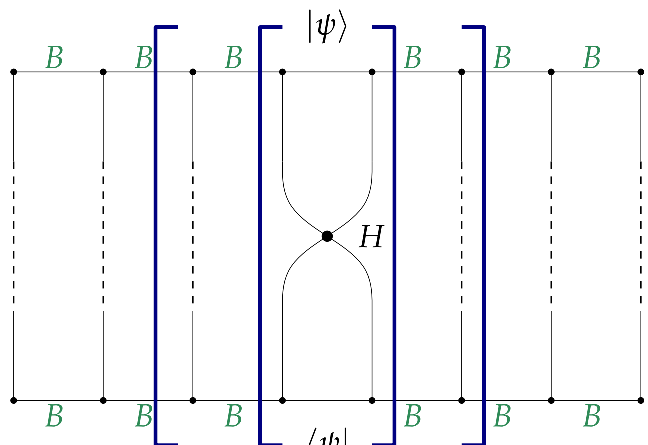

Figure 4: Computing the energy

8.2 The Detectibility lemma

We now discuss the *detectability lemma*, which provides the basis for a local constraint satisfaction view of the local Hamiltonian problem. More formally, given a d-local Hamiltonian $H = \sum_{i=1}^{n} H_i$ and a state $|\psi\rangle$ that is "far" from being its ground state in the sense of having a high value for the energy $\langle\psi|H|\psi\rangle$, we want to say that we will be able to "detect" the high energy simply by making the local measurements $H_1, H_2, \ldots, H_n$ in sequence.[1]

The detectability lemma [AALV09] provides some sufficient conditions under which this is possible. We let the H_i be *projections*, and assume that H is *frustration free*, i.e., there is a state $|\psi_0\rangle$ that is simultaneously a ground state of all the H_i (and hence has energy 0). We further assume that the Hamiltonian is *gapped*, i.e., there is a positive constant δ such that any state $|\phi\rangle$ orthogonal to $|\psi_0\rangle$ has energy at least δ: $\langle\phi|H|\phi\rangle \geq \delta$.

Ideally, we would like to prove that whenever ϕ is of the form $\langle\phi|H|\phi\rangle$, then the probability of failing to detect it using a sequential measurement as above is at most $1-\Omega(\delta)$, i.e.,

$$\|(I-H_m)(I-H_{m-1})\ldots(I-H_1)|\phi\rangle\|_2^2 = 1-\Omega(\delta).$$

However, this statement cannot be true. Suppose for example that there exists a state $|u\rangle$ such that $\langle u|H_i|u\rangle = 1$ for $1 \leq i \leq m$. Then, a simple calculation shows that the state $|\phi\rangle := \sqrt{1-\delta/n}|\psi_0\rangle + \sqrt{\delta/n}|u\rangle$ satisfies $\langle\phi|H|\phi\rangle = \delta$, even though the probability of failure is $1-\delta/n$. The actual detectability lemma gets around this by only considering states ϕ that are orthogonal to the ground state $|\psi_0\rangle$.

Lemma 1 (**Detectability lemma [AALV09]**)**.** *Let $H = \sum_{i=1}^n H_i$ be a frustration free Hamiltonian such that (i) all H_i are projections, and (ii) each H_i commutes with all but (at most) d other terms H_j. We further assume that $|\psi_0\rangle$ is the unique ground state of H with energy 0, and any state ψ orthogonal to $|\psi_0\rangle$ satisfies $\langle\psi|H|\psi\rangle \geq \delta > 0$. Then, for every $|\psi\rangle$ orthogonal to $|\psi_0\rangle$ we have*

$$\|(I-H_m)(I-H_{m-1})\ldots(I-H_1)|\psi\rangle\|^2 \leq 1-\Omega\left(\frac{\delta}{d^2}\right).$$

Proof. Let P_i be the projection to the ground state of H_i (thus, $H_i = I - P_i$, since H_i are projections). We define

$$|\phi\rangle := P_n P_{n-1}\ldots P_1|\psi\rangle.$$

Note that our goal is to show that $\||\phi\rangle\|^2 = 1-\Omega(\delta/d^2)$. We now consider the term $\langle\phi|H_i|\phi\rangle$. Observe that if $P_m, P_{m-1}, \ldots, P_{i-1}$ commute with H_i then $\langle\phi|H_i|\phi\rangle = 0$. So, let $i_1 > i_2 > \cdots > i_d$ be the (possible) indices greater than i such that H_{i_k}, $1 \leq k \leq d$ do not commute with H_i. We denote the set of these indices by $N(i)$. We then have

$$\begin{aligned}
H_i|\phi\rangle &= P_m\ldots P_{i_1+1}H_iP_{i_1-1}\ldots P_1|\psi\rangle - P_m\ldots P_{i_1+1}H_iH_{i_1}P_{i_1-1}\ldots P_1|\psi\rangle, \\
&\quad \text{using } I - H_{i_1} = P_{i_1}, \\
&= -\sum_{j\in N(i)}\left(\prod_{\substack{k>j \\ k\notin N(i)}} P_k\right) H_iH_jP_{j-1}\ldots P_1|\psi\rangle, \\
&\quad \text{by iterating the last step for each } i_k, 1 \leq k \leq d.
\end{aligned}$$

[1]The precise order of the measurements does not matter. The crucial point is that each of the n measurements involves only a small number of qubits, and is the non-commutative analog of classical tests of low query complexity.

Now, since H_i is a projection, we have

$$\begin{aligned}\langle\phi|H_i|\phi\rangle &= \| H_i|\phi\rangle\|^2 \\ &\leq d \sum_{j\in N(i)} \left\| H_j P_{j-1}\dots P_1|\psi\rangle\right\|^2, \\ &\qquad \text{using the Schwarz inequality, followed by neglecting some projections,} \\ &= \sum_{j\in N(i)} \left[\left\|P_{j-1}\dots P_1|\psi\rangle\right\|^2 - \left\|P_j P_{j-1}\dots P_1|\psi\rangle\right\|^2\right], \text{ since } H_j \text{ are projections.}\end{aligned}$$

Thus, summing over i, we have

$$\begin{aligned}\langle\phi|H|\phi\rangle &\leq d\sum_{i=1}^{n}\sum_{j\in N(i)} \left[\left\|P_{j-1}\dots P_1|\psi\rangle\right\|^2 - \left\|P_j P_{j-1}\dots P_1|\psi\rangle\right\|^2\right] \\ &\leq d^2 \sum_{j=1}^{n} \left(\left\|P_{j-1}\dots P_1|\psi\rangle\right\|^2 - \left\|P_j P_{j-1}\dots P_1|\psi\rangle\right\|^2\right), \\ &\qquad \text{since each of the terms can appear for at most } d \text{ values of } i, \\ &= d^2\left(1 - \|P_n\dots P_1|\psi\rangle\|^2\right) = d^2\left(1 - \||\phi\rangle\|^2\right).\end{aligned}$$

Now, we note that $|\phi\rangle$ is orthogonal to the ground state $|\psi_0\rangle$:

$$\langle\phi|\psi_0\rangle = \langle\psi|P_1\cdots P_n|\psi_0\rangle = \langle\psi|\psi_0\rangle = 0,$$

where for the second equality we used $P_i|\psi_0\rangle = |\psi_0\rangle$ for all i. Hence from the hypothesis of the theorem, we have $\langle\phi|H|\phi\rangle \geq \delta\,\||\phi\rangle\|^2$. Using the above inequality, we then have

$$\||\phi\rangle\|^2 \leq \frac{1}{1+\frac{\delta}{d^2}},$$

which completes the proof. □

References

[AALV09] Dorit Aharonov, Itai Arad, Zeph Landau, and Umesh Vazirani. The detectability lemma and quantum gap amplification. In *Proc. 41st ACM Symp. Theory Comput.*, STOC '09, pages 417–426, New York, NY, USA, 2009. ACM.