Before stating a quantum de Finetti theorem for density operators, we should define permutation invariance for quantum states. Let $S_n$ be the set of all permutations on $n$ elements taken from some finite set $[d] = \{1, \ldots, d\}$.

**Definition 1.** If $\pi$ is a permutation on $n$ elements from $[d]$, then the corresponding permutation operator on a state of $n$ qudits is defined as

$$ P_d(\pi) = \sum_{i_1, \ldots, i_n \in [d]} |i_{\pi(1)}, \ldots, i_{\pi(n)}\rangle \langle i_1, \ldots, i_n| $$

**Definition 2.** The symmetric subspace on $n$ qudits (a subspace of $(\mathbb{C}^d)^\otimes n$) is defined as

$$ V^n(C^d) = \text{span}\left\{ |\varphi\rangle \in (\mathbb{C}^d)^\otimes n \text{ s.t. } P_d(\pi)|\varphi\rangle = |\varphi\rangle \forall \pi \in S_n \right\} $$

**Definition 3.** An $n$-qudit quantum state $\rho \in \text{Dens}((\mathbb{C}^d)^\otimes n)$ is called $n$-exchangeable if it is invariant under the action of all the $P_d(\pi)$:

$$ P_d(\pi)\rho P_d(\pi) = \rho, \forall \pi \in S_n. $$

With these definitions in hand, we can now state a Quantum de Finetti theorem:

**Theorem 4.** Let $\rho \in \text{Dens}((\mathbb{C}^d)^\otimes n)$ be $n$-exchangeable. Then there exists a measure $\mu$ on $\text{Dens}(\mathbb{C}^d)$ such that

$$ \| \text{Tr}_{n-k}(\rho) - \int \sigma^\otimes k d\mu(\sigma) \|_1 \leq \frac{2k(d+k)}{n+d}, \forall k \leq n. $$

Remark: the $d$ dependence in the bound is necessary: for any $n$ there always exists a state $|\psi\rangle$ such that $\rho = |\psi\rangle \langle \psi|$ is permutation invariant on $(\mathbb{C}^d)^\otimes n$ and

$$ \| \text{Tr}_{n-2}(\rho) - \int \sigma^\otimes 2 d\mu(\sigma) \|_1 \geq \frac{1}{4} $$

for any measure $\mu$. Such a $\rho$ can be obtained from the projector onto the antisymmetric subspace:

$$ \rho = \frac{1}{n!} \sum_{\pi, \pi' \in S_n} \text{sgn}(\pi)\text{sgn}(\pi') |i_{\pi(1)}, \ldots, i_{\pi(n)}\rangle \langle i_{\pi'(1)}, \ldots, i_{\pi'(n)}| $$

Note also that for the special case of a pure state, there is a bound $2dk/n$ matching the classical bound (for $k$ large with respect to $d$).
# Proof of the Quantum de Finetti Theorem

We’ll need yet more definitions and notation allowing us to discuss the symmetric subspace in more detail.

**Definition 5.** A type is a vector of non-negative integers \( \vec{t} = (t_1, t_2, ..., t_d) \) such that \( \sum_{i=1}^{d} t_i = n \). We can think of a type \( \vec{t} \) as specifying how many 0s, 1s, 2s etc that a ket has. For example for \( n = 4, d = 3 \) the type \( \vec{t} = (3, 0, 1) \) specifies the kets \{\(|002\rangle, |020\rangle, |020\rangle, |200\rangle\} \). Each type \( \vec{t} \) can be thought of as specifying a permutation invariant vector in \( V^n (\mathbb{C}^d) \):

\[
|\vec{t}\rangle = \frac{1}{\#t} \sum |i_1, ..., i_n\rangle
\]  

where \( \#t \) is the number of kets that the type \( \vec{t} \) specifies, and the sum is over all such kets. That is, the sum is over \((i_1, ..., i_n) \in [d]^n \) such that \#\{i_j : i_j = 0\} = t_1, \#\{i_j : i_j = 1\} = t_2, etc...

**Claim 6.** \( \{|\vec{t}\rangle\} \) is a basis for \( V^n (\mathbb{C}^d) \).

**Corollary 7.** \( \dim (V^n (\mathbb{C}^d)) = \binom{n+d-1}{d-1} \), the number of distinct \( n \)-element types with values in \([d]\).

As a few concrete examples:

- For \( n = 2 \), \( \dim (V^2 (\mathbb{C}^d)) = \binom{d+1}{d-1} = \frac{d(d+1)}{2} \)

- For \( n = 2 = d \), \( \dim (V^2 (\mathbb{C}^2)) = 3 \) with basis \( \{|00\rangle, |11\rangle, \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)\} \) (notice this is like a basis for \( (\mathbb{C}^2)^{\otimes 2} \) but missing the antisymmetric state \( \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \))

**Claim 8.** The orthogonal projector onto \( V^n (\mathbb{C}^d) \) is

\[
P_{sym}^{d,n} = \frac{1}{n!} \sum_{\pi \in S_n} P_d(\pi).
\]  

**Proof.** First we show that \( P_{sym}^{d,n} \) is a projector:

\[
P_{sym}^{d,n} (P_{sym}^{d,n})^+ = \frac{1}{(n!)^2} \sum_{\pi, \pi' \in S_n} P_d(\pi) P_d(\pi')^+
\]

\[
= \frac{1}{(n!)^2} \sum_{\pi, \pi' \in S_n} P_d(\pi) P_d(\pi'^{-1})
\]

\[
= \frac{1}{(n!)^2} \sum_{\pi, \pi' \in S_n} P_d(\pi'^{-1} \pi)
\]

\[
= \frac{1}{(n!)^2} \sum_{\sigma \in S_n} n! P_d(\sigma)
\]

\[
= P_{sym}^{d,n}.
\]

Now, note that \( \text{Im} \left( P_{sym}^{d,n} \right) \subseteq V^n (\mathbb{C}^d) : \forall \pi \in S_n, P_d(\pi) P_{sym}^{d,n} = P_{sym}^{d,n} \). Also we have that \( \forall |\psi\rangle \in V^n (\mathbb{C}^d), P_{sym}^{d,n} |\psi\rangle = |\psi\rangle \implies V^n (\mathbb{C}^d) \subseteq \text{Im} \left( P_{sym}^{d,n} \right) \). Thus \( P_{sym}^{d,n} \) is the orthogonal projector onto \( V^n (\mathbb{C}^d) \).

\[\square\]
Claim 9.

\[
\frac{P_{\text{sym}}^{d,n}}{\Tr(P_{\text{sym}}^{d,n})} = \mathbb{E}_{|\varphi\rangle \in \mathcal{C}^d} |\varphi\rangle \langle \varphi |^{\otimes n} = \int_{\mathcal{C}^d} |\varphi\rangle \langle \varphi |^{\otimes n} d\mu (|\varphi\rangle)
\]

(9)

where \( \mu \) is the Haar measure on single qudits.

Proof. Exercise for the reader (Hint: use Schur’s lemma from representation theory).

\[\square\]

Definition 10. The \( n \rightarrow k \) ‘measure and prepare map’ \( \text{MP}_{n \rightarrow k} : \text{Dens} \left( (\mathbb{C}^d)^{\otimes n} \right) \rightarrow \text{Dens} \left( (\mathbb{C}^d)^{\otimes k} \right) \) is defined as

\[
\frac{d[n]}{d[n+k]} \text{MP}_{n \rightarrow k} (\rho) = \Tr_n \left( P_{\text{sym}}^{d,n+k} \left( \rho \otimes \mathbb{I}^{\otimes k} \right) \right),
\]

(10)

where we introduced the notation \( d[n] = \binom{n+d-1}{d-1} \) for the dimension of \( V^n (\mathbb{C}^d) \). Intuitively what this map does is measure in a random basis and prepare \( k \) copies of the outcome.

An important observation is that the range of \( \text{MP}_{n \rightarrow k} \) lies in the convex hull of all tensor product states.

This is apparent by using the formulation for \( P_{\text{sym}}^{d,n+k} \) given in Claim 9 to write

\[
\text{MP}_{n \rightarrow k} (\rho) = \Tr_n \left( \int_{\mathcal{C}^d} |\varphi\rangle \langle \varphi |^{\otimes (n+k)} d\mu (|\varphi\rangle) (\rho \otimes \mathbb{I}) \right) = \int_{\mathcal{C}^d} \langle \varphi |^{\otimes n} \rho |\varphi\rangle^{\otimes n} |\varphi\rangle \langle \varphi |^{\otimes k} d\mu (|\varphi\rangle).
\]

Definition 11. The ‘optimal cloning map’ \( \text{Cl}_{n \rightarrow n+k} : \text{Dens} \left( (\mathbb{C}^d)^{\otimes n} \right) \rightarrow \text{Dens} \left( (\mathbb{C}^d)^{\otimes n+k} \right) \) is defined as

\[
\text{Cl}_{n \rightarrow n+k} (\rho) = \frac{d[n]}{d[n+k]} P_{\text{sym}}^{d,n+k} \left( \rho \otimes \mathbb{I}^{\otimes k} \right) P_{\text{sym}}^{d,n+k}.
\]

(11)

Intuitively, what this does is create the symmetric state on \( n+k \) qudits closest to the input state (comes as close to symmetrically 'cloning' as possible). One can show that it is trace preserving.

Claim 12. \( \text{MP}_{n \rightarrow k} (\rho) = \sum_{s=0}^{k} \binom{n}{s} \binom{n+k-1}{s} \text{Cl}_{n \rightarrow s} (\Tr_{n-s} (\rho)) \)

Proof. The proof of this claim is left as a non-trivial exercise; it’s a lengthy calculation that takes advantage of the fact that vectors of the form \( |\varphi\rangle \langle \varphi |^{\otimes n} \) span the set of all permutation-invariant density matrices (though the combinations may require arbitrary complex coefficients) to reduce the calculation to comparing the action of each maps on those state.

\[\square\]

We can see that the theorem follows easily from this claim. Indeed, for \( s = k \) we have for the coefficient

\[
\binom{n}{k} \binom{n+k-1}{k} = \frac{n(n-1)\ldots(n-k+1)}{(d+n+k-1)\ldots(d+n)} \]

(12)

\[
\geq \left( \frac{n-k+1}{d+n} \right)^k
\]

(13)

\[
= \left( 1 - \frac{d+k-1}{d+n} \right)^k
\]

(14)

\[
\geq 1 - \frac{k(d+k-1)}{d+n}.
\]

(15)
Definition 13. A POVM \( \mathcal{M} \) is called informationally complete with distortion \( \gamma \) if
\[
\sup_{X \in \mathbb{C}^{d \times d}, \, \text{s.t.} \, X \neq 0, \, \text{Tr}(X) = 0} \frac{\|X\|_{\text{Tr}}}{\|\mathcal{M}(X)\|_{\text{Tr}}} \leq \gamma < \infty
\] (20)

Some examples

- \( \mathcal{M} = \{ \langle 0 | 0 \rangle, \langle 1 | 1 \rangle \} \) is NOT informationally complete, note that \( X = \langle + | + \rangle - \langle - | - \rangle \) is such that \( \mathcal{M}(X) = 0 \) but \( \|X\|_{\text{Tr}} > 0 \).

- \( \mathcal{M} = \{ \frac{1}{2} \langle 0 | 0 \rangle, \frac{1}{2} \langle 1 | 1 \rangle, \frac{1}{2} \langle + | + \rangle, \frac{1}{2} \langle - | - \rangle, \frac{1}{2} | i \rangle \langle i |, \frac{1}{2} | - i \rangle \langle - i | \} \), where \( | \pm i \rangle = \frac{1}{\sqrt{2}} (|0 \rangle \pm i |1 \rangle) \), is informationally complete: If \( \langle 0 | X | 0 \rangle = \langle 1 | X | 1 \rangle = 0 \) then \( \langle \frac{3}{8} | X | \frac{3}{8} \rangle = \frac{1}{2} (X_{12} + X_{21}) \) and \( \langle i | X | i \rangle = i (X_{12} - X_{21}) \), \( \implies X = 0 \) if these are 0.

- Exercise: give an informationally complete measurement on \( \mathbb{C}^2 \) with only four possible outcomes. Show that this is best possible.
Claim 14. For every $d$ there exists a measurement $\Lambda$ with $s \leq d^8$ outcomes such that for every $k$, $\Lambda^k$ is an informationally complete measurement on $(\mathbb{C}^d)^\otimes k$ with distortion $\gamma \leq (18d)^{k/2}$.

Proof. One can give a probabilistic argument; $\Lambda$ is sum of $d^8$ random rank-one projectors. The argument is fairly easy if $k = 1$ (concentration + union bound), but harder for general $k$: the nontrivial fact is that the same $\Lambda$ will work for arbitrary $k$.

Note: informationally complete measurements are related to tomography: how many measurements does one need to make in order to uniquely identify a state (which we have infinite copies of)?

We can now state a second quantum de Finetti theorem, which is incomparable to the previous one:

Theorem 15. Let $\rho \in \text{Dens} \left( (\mathbb{C}^2)^\otimes n \right)$, $k \leq n$, and $t \leq n - k$. Then there exists an $m \leq t$ such that if for any $i \in [n]^k$, $j \in [n]^m$ and $x \in [d^8]^m$ we let $\rho_{i,j,x}$ be the post measurement state on qubits $i_1, \ldots, i_n$ obtained after measuring $j_1, \ldots, j_m$ using $\Lambda$ from the above claim and obtaining outcomes $x_1, \ldots, x_m$, then we have that

$$E_{i,j,x} \| \rho_{i,j,x} - \rho_{i_1,j_1,x_1} \otimes \cdots \otimes \rho_{i_k,j_k,x_k} \|_1^2 \leq \frac{2 \ln d (18d)^{k^2}}{t}.$$  (21)