CS286.2 Lecture 13: Quantum de Finetti Theorems

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Before stating a quantum de Finetti theorem for density operators, we should define permutation invariance for quantum states. Let S_n be the set of all permutations on n elements taken from some finite set $[d] = \{1, \ldots, d\}$.

Definition 1. *If π is a permutation on n elements from $[d]$, then the corresponding permutation operator on a state of n qudits is defined as*

$$P_d(\pi) = \sum_{i_1,\ldots,i_n \in [d]} |i_{\pi(1)}, \ldots, i_{\pi(n)}\rangle\langle i_1, \ldots, i_n| \tag{1}$$

Definition 2. *The symmetric subspace on n qudits (a subspace of $\left(\mathbb{C}^d\right)^{\otimes n}$) is defined as*

$$V^n\left(\mathbb{C}^d\right) = \text{span}\left\{|\varphi\rangle \in \left(\mathbb{C}^d\right)^{\otimes n} \; s.t. \; P_d(\pi)|\varphi\rangle = |\varphi\rangle \; \forall \, \pi \in S_n\right\} \tag{2}$$

Definition 3. *An n-qudit quantum state $\rho \in \text{Dens}\left(\left(\mathbb{C}^d\right)^{\otimes n}\right)$ is called n-exchangeable if it is invariant under the action of all the $P_d(\pi)$:*

$$P_d(\pi)\rho P_d(\pi) = \rho, \; \forall \, \pi \in S_n. \tag{3}$$

With these definitions in hand, we can now state a Quantum de Finetti theorem:

Theorem 4. *Let $\rho \in \text{Dens}\left(\left(\mathbb{C}^d\right)^{\otimes n}\right)$ be n-exchangeable. Then there exists a measure μ on $\text{Dens}\left(\mathbb{C}^d\right)$ such that*

$$\|\text{Tr}_{n-k}(\rho) - \int \sigma^{\otimes k} d\mu(\sigma)\|_1 \leq \frac{2k(d+k)}{n+d}, \; \forall \, k \leq n. \tag{4}$$

Remark: the d dependence in the bound is necessary: for any n there always exists a state $|\psi\rangle$ such that $\rho = |\psi\rangle\langle\psi|$ is permutation invariant on $\left(\mathbb{C}^d\right)^{\otimes n}$ and

$$\|\text{Tr}_{n-2}(\rho) - \int \sigma^{\otimes 2} d\mu(\sigma)\|_1 \geq \frac{1}{4} \tag{5}$$

for *any* measure μ. Such a ρ can be obtained from the projector onto the *anti*symmetric subspace:

$$\rho = \frac{1}{n!} \sum_{\pi, \pi' \in S_n} \text{sgn}(\pi)\text{sgn}(\pi')|i_{\pi(1)}, \ldots, i_{\pi(n)}\rangle\langle i_{\pi'(1)}, \ldots, i_{\pi'(n)}| \tag{6}$$

Note also that for the special case of a pure state, there is a bound $2dk/n$ matching the classical bound (for k large with respect to d).

Proof of the Quantum de Finetti Theorem

We'll need yet more definitions and notation allowing us to discuss the symmetric subspace in more detail

Definition 5. *A type is a vector of non-negative integers* $\vec{t} = (t_1, t_2, ..., t_d)$ *such that* $\sum_{i=1}^{d} t_i = n$*. We can think of a type* $\vec{t}$ *as specifying how many* 0*s,* 1*s,* 2*s etc that a ket has. For example for* $n = 4$*,* $d = 3$ *the type* $\vec{t} = (3, 0, 1)$ *specifies the kets* $\{|0002\rangle, |0200\rangle, |0020\rangle, |2000\rangle\}$*. Each type* $\vec{t}$ *can be thought of as specifying a permutation invariant vector in* $V^n\left(\mathbb{C}^d\right)$*:*

$$|\vec{t}\rangle = \frac{1}{\sqrt{\#\vec{t}}} \sum |i_1, ..., i_n\rangle \tag{7}$$

where $\#\vec{t}$ *is the number of kets that the type* $\vec{t}$ *specifies, and the sum is over all such kets. That is, the sum is over* $(i_1, ..., i_n) \in [d]^n$ *such that* $\#\{i_j : i_j = 0\} = t_1$*,* $\#\{i_j : i_j = 1\} = t_2$*, etc...*

Claim 6. $\{|\vec{t}\rangle\}$ *is a basis for* $V^n\left(\mathbb{C}^d\right)$*.*

Corollary 7. $\dim\left(V^n\left(\mathbb{C}^d\right)\right) = \binom{n+d-1}{d-1}$*, the number of distinct* n*-element types with values in* $[d]$*.*

As a few concrete examples:

- For $n = 2$, $\dim\left(V^2\left(\mathbb{C}^d\right)\right) = \binom{d+1}{d-1} = \frac{d(d+1)}{2}$
- For $n = 2 = d$, $\dim\left(V^2\left(\mathbb{C}^2\right)\right) = 3$ with basis $\{|00\rangle, |11\rangle, \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)\}$ (notice this is like a basis for $\left(\mathbb{C}^2\right)^{\otimes 2}$ but missing the antisymmetric state $\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$)

Claim 8. *The orthogonal projector onto* $V^n\left(\mathbb{C}^d\right)$ *is*

$$P_{sym}^{d,n} = \frac{1}{n!} \sum_{\pi \in S_n} P_d(\pi). \tag{8}$$

Proof. First we show that $P_{sym}^{d,n}$ is a projector:

$$\begin{aligned}
P_{sym}^{d,n}\left(P_{sym}^{d,n}\right)^\dagger &= \frac{1}{(n!)^2} \sum_{\pi, \pi' \in S_n} P_d(\pi) P_d(\pi')^\dagger \\
&= \frac{1}{(n!)^2} \sum_{\pi, \pi' \in S_n} P_d(\pi) P_d(\pi'^{-1}) \\
&= \frac{1}{(n!)^2} \sum_{\pi, \pi' \in S_n} P_d(\pi'^{-1}\pi) \\
&= \frac{1}{(n!)^2} \sum_{\sigma \in S_n} n! P_d(\sigma) \\
&= P_{sym}^{d,n}.
\end{aligned}$$

Now, note that $\operatorname{Im}\left(P_{sym}^{d,n}\right) \subseteq V^n\left(\mathbb{C}^d\right) : \forall\, \pi \in S_n,\ P_d(\pi) P_{sym}^{d,n} = P_{sym}^{d,n}$. Also we have that $\forall\, |\psi\rangle \in V^n\left(\mathbb{C}^d\right),\ P_{sym}^{d,n}|\psi\rangle = |\psi\rangle \implies V^n\left(\mathbb{C}^d\right) \subseteq \operatorname{Im}\left(P_{sym}^{d,n}\right)$. Thus $P_{sym}^{d,n}$ is the orthogonal projector onto $V^n\left(\mathbb{C}^d\right)$. $\square$

Claim 9.

$$\frac{P_{sym}^{d,n}}{\operatorname{Tr}\left(P_{sym}^{d,n}\right)} = \mathbb{E}_{|\varphi\rangle\in\mathbb{C}^d}|\varphi\rangle\langle\varphi|^{\otimes n} = \int_{\mathbb{C}^d}|\varphi\rangle\langle\varphi|^{\otimes n}d\mu\left(|\varphi\rangle\right) \tag{9}$$

where μ is the Haar measure on single qudits.

Proof. Exercise for the reader (Hint: use Schur's lemma from representation theory). □

Definition 10. *The $n \to k$ 'measure and prepare map' $\mathrm{MP}_{n\to k}$: Dens $\left(\left(\mathbb{C}^d\right)^{\otimes n}\right) \to$ Dens $\left(\left(\mathbb{C}^d\right)^{\otimes k}\right)$ is defined as*

$$\frac{d[n]}{d[n+k]}\mathrm{MP}_{n\to k}\left(\rho\right) = \operatorname{Tr}_n\left(P_{sym}^{d,n+k}\left(\rho\otimes I^{\otimes k}\right)\right), \tag{10}$$

where we introduced the notation $d[n] = \binom{n+d-1}{d-1}$ for the dimension of $V^n\left(\mathbb{C}^d\right)$. Intuitively what this map does is measure in a random basis and prepare k copies of the outcome.

An important observation is that the range of $\mathrm{MP}_{n\to k}$ lies in the convex hull of all tensor product states. This is apparent by using the formulation for $P_{sym}^{d,n+k}$ given in Claim 9 to write

$$\mathrm{MP}_{n\to k}(\rho) = \operatorname{Tr}_n\left(\int_{\mathbb{C}^d}|\varphi\rangle\langle\varphi|^{\otimes(n+k)}d\mu\left(|\varphi\rangle\right)\left(\rho\otimes\mathrm{Id}\right)\right) = \int_{\mathbb{C}^d}\langle\varphi|^{\otimes n}\rho|\varphi\rangle^{\otimes n}|\varphi\rangle\langle\varphi|^{\otimes k}d\mu\left(|\varphi\rangle\right).$$

Definition 11. *The 'optimal cloning map' $\mathrm{Cl}_{n\to n+k}$: Dens $\left(\left(\mathbb{C}^d\right)^{\otimes n}\right) \to$ Dens $\left(\left(\mathbb{C}^d\right)^{\otimes n+k}\right)$ is defined as*

$$\mathrm{Cl}_{n\to n+k}\left(\rho\right) = \frac{d[n]}{d[n+k]}P_{sym}^{d,n+k}\left(\rho\otimes\mathrm{Id}^{\otimes k}\right)P_{sym}^{d,n+k}. \tag{11}$$

Intuitively, what this does is create the symmetric state on $n+k$ qudits closest to the input state (comes as close to symmetrically 'cloning' as possible). One can show that it is trace preserving.

Claim 12. $\mathrm{MP}_{n\to k}\left(\rho\right) = \sum_{s=0}^{k}\frac{\binom{n}{s}\binom{d+k-1}{k-s}}{\binom{d+k+n-1}{k}}\mathrm{Cl}_{s\to k}\left(\operatorname{Tr}_{n-s}\left(\rho\right)\right)$

Proof. The proof of this claim is left as a non-trivial exercise; it's a lengthy calculation that takes advantage of the fact that vectors of the form $|\phi\rangle\langle\phi|^{\otimes n}$ span the set of all permutation-invariant density matrices (though the combinations may require arbitrary complex coefficients) to reduce the calculation to comparing the action of each maps on those state. □

We can see that the theorem follows easily from this claim. Indeed, for $s = k$ we have for the coefficient

$$\begin{aligned}\frac{\binom{n}{k}}{\binom{d+n+k-1}{k}} &= \frac{n(n-1)...(n-k+1)}{(d+n+k-1)...(d+n)} &(12)\\ &\geq \left(\frac{n-k+1}{d+n}\right)^k &(13)\\ &= \left(1-\frac{d+k-1}{d+n}\right)^k &(14)\\ &\geq 1-\frac{k(d+k-1)}{d+n}. &(15)\end{aligned}$$

So, using the triangle inequality we have that

$$\|\mathrm{MP}_{n\to k}(\rho) - \mathrm{Cl}_{k\to k}(\mathrm{Tr}_{n-k}(\rho))\|_1 \leq \frac{k(d+k-1)}{d+n}\|\mathrm{Cl}_{k\to k}(\mathrm{Tr}_{n-k}(\rho))\|_1 + \sum_{s=0}^{k-1}\frac{(...)(...)}{(...)}\|\mathrm{Cl}_{s\to k}(\mathrm{Tr}_{n-s}(\rho))\|_1 \tag{16}$$

and using that $\|\mathrm{Cl}_{s\to k}(\mathrm{Tr}_{n-s}(\rho))\|_1 = 1$ and that the sum of the coefficients of the second term are bounded by $\frac{k(d+k-1)}{d+n}$, we have that

$$\|\mathrm{MP}_{n\to k}(\rho) - \mathrm{Cl}_{k\to k}(\mathrm{Tr}_{n-k}(\rho))\|_1 \leq \frac{2k(d+k-1)}{d+n}. \tag{17}$$

So, realizing that $\mathrm{Cl}_{k\to k}(\mathrm{Tr}_{n-k}(\rho)) = \mathrm{Tr}_{n-k}(\rho)$ and recalling that the measure and prepare map outputs a convex combination of product states, the proof of our first quantum de Finetti theorem, Theorem 4, is complete.

Informationally Complete Measurements and a second Quantum de Finetti Theorem

Let $\mathcal{M} = \{M_0, ..., M_{s-1}\}$ be a general POVM on $\mathbb{C}^d$ and recall the measurement rule for density matrices: if $\rho \in \mathrm{Dens}\left(\mathbb{C}^d\right)$ is subject to measurement described by the POVM $\mathcal{M}$, then the Born rule says that probability of obtaining outcome M_i is $p_i = \mathrm{Tr}\left(M_i\rho\right)$. So, to $\mathcal{M}$ we can actually associate a map

$$\mathcal{M} : \mathrm{Dens}\left(\mathbb{C}^d\right) \to \{\text{distributions on d}\} \tag{18}$$

defined as

$$\mathcal{M}(\rho) = \sum_{i=0}^{s-1}\mathrm{Tr}\left(M_i\rho\right)|i\rangle\langle i|, \tag{19}$$

where the output is actually a diagonal density matrix with the probabilities p_i on the diagonal.

Definition 13. *A POVM $\mathcal{M}$ is called informationally complete with distortion γ if*

$$\sup_{X\in\mathcal{C}^{d\times d}\ s.t.\ X\neq 0,\mathrm{Tr}(X)=0}\frac{\|X\|_{Tr}}{\|\mathcal{M}(X)\|_{Tr}}. \leq \gamma < \infty \tag{20}$$

Some examples

- $\mathcal{M} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ is NOT informationally complete, note that $X = |+\rangle\langle +| - |-\rangle\langle -|$ is such that $\mathcal{M}(X) = 0$ but $\|X\|_{Tr} > 0$.
- $\mathcal{M} = \{\frac{1}{3}|0\rangle\langle 0|, \frac{1}{3}|1\rangle\langle 1|, \frac{1}{3}|+\rangle\langle +|, \frac{1}{3}|-\rangle\langle -|, \frac{1}{3}|i\rangle\langle i|, \frac{1}{3}|-i\rangle\langle -i|\}$, where $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle$, is informationally complete: If $\langle 0|X|0\rangle = \langle 1|X|1\rangle = 0$ then $\langle\frac{\pi}{8}|X|\frac{\pi}{8}\rangle = \frac{1}{2}(X_{12}+X_{21})$ and $\langle i|X|i\rangle = i(X_{12}-X_{21}) \implies X = 0$ if these are 0.
- Exercise: give an informationally complete measurement on $\mathbb{C}^2$ with only four possible outcomes. Show that this is best possible.

Claim 14. *For every d there exists a measurement Λ with $s \leq d^8$ outcomes such that for every k, $\Lambda^{\otimes k}$ is an informationally complete measurement on $\left(\mathbb{C}^d\right)^{\otimes k}$ with distortion $\gamma \leq (18d)^{k/2}$.*

Proof. One can give a probabilistic argument; Λ is sum of d^8 random rank-one projectors. The argument is fairly easy if $k = 1$ (concentration + union bound), but harder for general k: the nontrivial fact is that the same Λ will work for arbitrary k.

Note: informationally complete measurements are related to tomography: how many measurements does one need to make in order to uniquely identify a state (which we have infinite copies of)? □

We can now state a second quantum de Finetti theorem, which is incomparable to the previous one:

Theorem 15. *Let $\rho \in \mathrm{Dens}\left(\left(\mathbb{C}^2\right)^{\otimes n}\right)$, $k \leq n$, and $t \leq n - k$. Then there exists an $m \leq t$ such that if for any $\vec{i} \in [n]^k$, $\vec{j} \in [n]^m$ and $\vec{x} \in [d^8]^m$ we let $\rho_{\vec{i}}^{\vec{j},\vec{x}}$ be the post measurement state on qubits $i_1, ..., i_n$ obtained after measuring $j_1, ..., j_m$ using Λ from the above claim and obtaining outcomes $x_1, ..., x_m$, then we have that*

$$\mathbb{E}_{\vec{i}}\mathbb{E}_{\vec{j},\vec{x}} \|\rho_{\vec{i}}^{\vec{j},\vec{x}} - \rho_{i_1}^{\vec{j},\vec{x}} \otimes ... \otimes \rho_{i_k}^{\vec{j},\vec{x}}\|_1^2 \leq \frac{2 \ln d (18d)^k k^2}{t}. \tag{21}$$