CS286.2 Lecture 11: 1-D Area Law

Scribe: Fernando Granha Jeronimo

In this lecture, we continue the proof of the area law for 1-D gapped Hamiltonian $H = \sum_{i=1}^{n-1} H_i$. We quickly remember the assumptions from last time. The Hamiltonian H acts on n qubits in a line where each term $0 \leq H_i \leq 1$ is a projection ($H_i^2 = H_i$) and it acts on qubits i and $i+1$. The ground state $|\psi_0\rangle$ is unique and H is frustration free ($\lambda_0(H) = 0$). Moreover, the spectral gap is a constant $\delta = (\lambda_1(H) - \lambda_0(H)) > 0$. This proof is based on the results of [1] and [2].

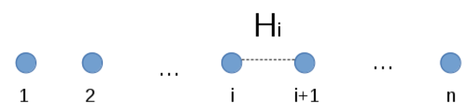

Figure 1: 1-D Chain

The central tool in our approach for the area law is the Approximate Ground State Projector (AGSP) operator relative to a certain cut of qubits $(A, \bar{A})$. Since we are working in 1-D, a cut can be specified by the pair $(i^\star, i^\star + 1)$ where $i^\star$ indexes a qubit in the chain (see Figure 1). When applied to a quantum state, the AGSP operator lets the ground state $|\psi_0\rangle$ invariant while shrinking components orthogonal to it. One undesired side-effect of this operation is that it increases the bond dimension across the associated cut. These two properties of the AGSP are captured by the parameters Δ and B whose precise definition is given next.

Definition 1 (AGSP)**.** *Given a Hamiltonian H, an (B, Δ)-AGSP for H with respect to the cut of qubits $i^\star$ and $i^\star + 1$ is an operator $K \in (\mathbb{C}^2)^{\otimes n}$ such that*

- $K|\psi_0\rangle = |\psi_0\rangle$;
- $\forall |\psi\rangle : \langle\psi|\psi_0\rangle = 0$, $\| K|\psi\rangle \|^2 \leq \Delta \| |\psi\rangle \|^2$;
- *In terms of tensor network representation, the bond dimension of state $K|\psi\rangle$ is at most B times its value for state $|\psi\rangle$ across the cut $i^\star$ and $i^\star + 1$.*

Note that we work with the bond dimension or the Schmidt Rank (SR) parameter (the number of nonzero Schmidt coefficients in the Schmidt decomposition across the cut), since it is simple to bound and it can latter help bounding the von Neumann entropy across the cut $i^\star$ and $i^\star + 1$. At each application of an AGSP, we may get closer to the ground state, but we may also increase the entropy. The appropriate trade-off between B and Δ satisfying $B\Delta \leq \frac{1}{2}$ captures the right balance between the rate of convergence to the ground state and the rate of increase in the entropy that is necessary to show the area law.

Theorem 2. *Suppose there exists a (B, Δ)-AGSP such that $B\Delta \leq \frac{1}{2}$. Then $|\psi_0\rangle$ satisfies an area law of the form $S(A)_{|\psi_0\rangle} \leq O(1) \log B$.*

In our 1-D case, the boundary of any bi-partition comprises a constant number of qubits, thus the area law states that the entanglement is bounded by a constant independent of the number of qubits n. For a constant B and the appropriate AGSP, the previous theorem implies the 1-D area law.

Theorem 3 (1-D Area Law)**.** *$|\psi_0\rangle$ satisfies an area law of the form $S(A)_{|\psi_0\rangle} \leq O(1)$.*

To prove Theorem 2, we first show the existence of a product state that has a constant overlap (depending on B only, and not on n) with the ground state. Then, starting from this state we successively apply the AGSP to get better and better approximations of the ground state while not increasing the bond dimension too much. The next lemma proved in the previous lecture achieves the first goal.

Lemma 4. *Suppose $\exists(B,\Delta)$-AGSP such that $B\Delta \leq \frac{1}{2}$. Fix a partition $(A,\bar{A})$ of the space on which the Hamiltonian acts. Then there exists a product state $|\phi\rangle = |L\rangle_A \otimes |R\rangle_{\bar{A}}$ such that $|\langle\phi|\psi_0\rangle| = \mu \geq \frac{1}{\sqrt{2B}}$.*

Before we move to the next lemma, we need an auxiliary result known as Eckart-Young Theorem. This result bounds the maximum possible overlap between states $|\psi\rangle$ and $|\phi\rangle$ where the latter has a predefined Schmidt Rank.

Theorem 5 (Eckart-Young)**.** *Let $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ be a normalized vector with Schmidt decomposition $|\psi\rangle = \sum \lambda_i |u_i\rangle|v_i\rangle$, where $\lambda_1 \geq \lambda_2 \geq \cdots$. Then for any normalized $|\phi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ with Schmidt rank at most B it holds that*

$$|\langle\psi|\phi\rangle| \leq \sqrt{\sum_{i=1}^{B} \lambda_i^2}.$$

Given a product state that has overlap μ with the ground state, the next lemma uses an AGSP to bound the entanglement across the associated cut $(i^\star, i^\star+1)$.

Lemma 6. *Suppose $\exists(B,\Delta)$-AGSP and $\exists$ a product state $|\phi\rangle = |L\rangle_A \otimes |R\rangle_{\bar{A}}$ such that $|\langle\phi|\psi_0\rangle| = \mu$. Then*

$$S((i^\star, i^\star+1))_{|\psi_0\rangle} \leq O(1)\frac{\log\mu}{\log\Delta}\log B \tag{1}$$

Proof. Denote by K the (B,Δ)-AGSP.Define $|\phi_l\rangle = \frac{K^l|\phi\rangle}{\|K^l|\phi\rangle\|}$. It is not hard to see that $|\phi_l\rangle$ is such that

(i) $SR(|\phi_l\rangle) \leq B^l$;

(ii) $|\langle\psi_0|\phi_l\rangle| \geq \frac{\mu}{\sqrt{\mu^2+\Delta^l(1-\mu^2)}}$.

Property (i) follows from $SR(|\phi\rangle) = 1$ and the AGSP having bond parameter B. Furthermore, property (ii) follows from the definition of the shrinking parameter Δ.

Let $|\psi_0\rangle = \sum \lambda_i |L_i\rangle|R_i\rangle$ be the Schmidt decomposition of the ground state relative to cut $(i^\star, i^\star+1)$, where $\lambda_1 \geq \lambda_2 \geq \ldots$. By the Eckart-Young Theorem we have

$$\sum_{i=1}^{B^l} \lambda_i^2 \geq |\langle\psi_0|\phi_l\rangle|^2 \geq \frac{\mu^2}{\mu^2+\Delta^l(1-\mu^2)}$$

or equivalently

$$\sum_{i>B^l} \lambda_i^2 \le 1 - \frac{\mu^2}{\mu^2 + \Delta^l(1-\mu^2)} \le 1 - \frac{\mu^2}{\mu^2+\Delta^l} \le \frac{\Delta^l}{\mu^2}.$$

We choose $l_0 = 2\frac{\log \mu}{\log \Delta} - \frac{\log 2}{\log \Delta}$ such that $\frac{\Delta^{l_0}}{\mu^2} \le \frac{1}{2}$ and proceed to bound the worst case entropy across the AGSP cut. The first B^{l_0} Schmidt coefficients account for an entropy of at most $l_0 \log B$. For the remaining coefficients, we group them in chunks of size B^{l_0} in intervals $[B^{kl_0}+1, B^{(k+1)l_0}]$ indexed by k. For each of these intervals, the corresponding entropy can be upper bounded by

$$\frac{\Delta^{kl_0}}{\mu^2} \log B^{(k+1)l_0} = l_0 \frac{1}{2^k}(k+1)\log B,$$

where here $\frac{\Delta^{kl_0}}{\mu^2}$ is an upper bound on the total probability mass in the interval, and $B^{-(k+1)l_0}$ a lower bound on the size of any Schmidt coefficient (squared) in the interval, which follows from the fact that they are organized in descending order and must sum to 1. Therefore, the total entropy is

$$\begin{aligned} S((i^\star, i^\star+1)) &\le l_0 \log B + \sum_{k\ge 1} l_0 \frac{1}{2^k}(k+1)\log B \\ &\le l_0 \log B + l_0 \log B \sum_{k \ge 1} \frac{1}{2^k}(k+1) \\ &\le O(1) l_0 \log B \\ &\le O(1) \frac{\log \mu}{\log \Delta} \log B \end{aligned}$$

We note that this bound is slightly weaker then the one that appears in [2]; nevertheless it is sufficient to imply the area law as our AGSP construction will have constant Δ and B (and hence μ). □

The Detectability Lemma gave us an AGSP with B = 4 and $\Delta = 1 - \Omega(\delta)$. Unfortunately, the product $B\Delta \approx 4$ when the gap δ is small does not satisfy our requirement. The proof of the next theorem shows that there is indeed an AGSP with the desired properties.

Theorem 7. *There is an (B, Δ)-AGSP with $B\Delta \le \frac{1}{2}$.*

To prove this theorem, we first claim some properties about Chebyshev polynomials and we apply them to the Hamiltonian H to construct the appropriate AGSP.

One important property that we make use is the subadditivity of the Schmidt Rank as stated in the following claim. After applying a polynomial to our Hamiltonian $H = \sum_{i=1}^{n-1} H_i$, we end up with a sum of products. The subadditivity implies that the Schmidt Rank of the sum is bounded by the sum of the Schmidt Ranks of these products.

Claim 8. *$SR(|\psi\rangle + |\phi\rangle) \le SR(|\psi\rangle) + SR(|\phi\rangle)$.*

Using Chebyshev polynomials, it is possible to construct a polynomial that maps 0 to 1 and shrinks values in a given interval.

Claim 9. *For any integer l, there exists a real polynomial P_l of degree l such that*

- $P_l(0) = 1$;
- $\forall x \in [\delta, \lambda^\star]$, $|P_l(x)| \leq \sqrt{\Delta} = 2e^{-2l\sqrt{\frac{\delta}{\lambda^\star}}}$.

A polynomial satisfying these conditions will look something like the representation in Figure 2.

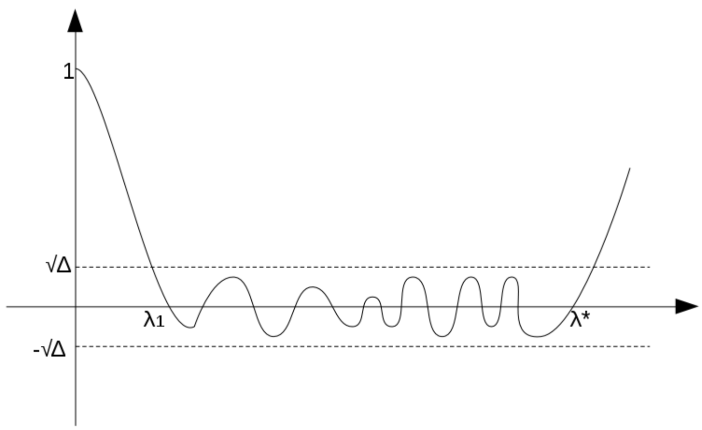

Figure 2: Sketch of a Chebyshev Polynomial

If we apply this polynomial to our Hamiltonian H that is Hermitian, it is equivalent to apply the polynomial to its eigenvalues in which case we obtain an operator $K_l = P_l(H)$ satisfying

(i) $K_l|\psi_0\rangle = |\psi_0\rangle$ since $P_l(0) = 1$;

(ii) if $|\langle\psi|\psi_0\rangle| = 0$, it has eigenvalue between $[\delta, \lambda^\star]$ where δ is the gap and $\lambda^\star$ is the operator norm of H and $\|K_l|\psi\rangle\| \leq \Delta$;

(iii) $SR(K_l) \leq (l+1)n^l 4^l$.

These properties together show that $P_l(H)$ is an $(B = (l+1)n^l 4^l, \Delta = 4e^{-4l\sqrt{\frac{\delta}{\lambda^\star}}})$-AGSP. Note that the bound (iii) stating that $SR(K_l) \leq (l+1)n^l 4^l$can be obtained in a very naive way: since P_l has degree l, by expanding out all monomials in $P_l(H_1 + \cdots + H_{n-1})$ we obtain at most $(l+1)n^l$ terms, each of which has Schmidt Rank bounded by 4^l. The bound follows using subadditivity of the Schmidt rank. Unfortunately this bound is not good enough to give us the area law, as B should not depend on n. Fortunately, there is a more clever way to bound B that results in $B\Delta \leq \frac{1}{2}$ as we explore next.

The first step is to replace H by

$$\tilde{H} = H_L + H_{i-\frac{u}{2}} + \cdots + H_i + \cdots + H_{i+\frac{u}{2}-1} + H_R,$$

where $H_L = trunc(H_1 + \cdots + H_{i-\frac{u}{2}-1}, t)$ and $H_R = trunc(H_{i+\frac{u}{2}} + \cdots + H_{n-1}, t)$. The operation $trunc(X, t)$, when applied to an Hermitian operator X, replaces all its eigenvalues larger than t by t. Therefore, it is clear that the operator norm of $\tilde{H}$ is $\lambda^\star \leq u + 2t$. Moreover, this process has not changed the ground state. A non-trivial observation is that the $\tilde{\delta} = \lambda_1(\tilde{H}) \geq \Omega(\lambda_1(H) = \delta)$ as long as t is chosen so that $t = O(\frac{1}{\delta})$. We will not prove this fact here, and for now on simply assume t to be fixed so that the spectral gap of $\tilde{H}$ is at least a constant times that of H. The key lemma that remains to be proven is the following bound on B.

Lemma 10. *Assume $\sqrt{l} \geq u$. Then $K = P_l(\tilde{H})$ is a (B, Δ)-AGSP with $\Delta \leq 4e^{-4l\sqrt{\frac{\tilde{\delta}}{\lambda^\star}}}$ and $B \leq (2l)^{O(\frac{l}{u})}$.*

Let's first verify that with the correct choice of parameters this lemma can actually lead to $B\Delta \leq \frac{1}{2}$, and hence establish the area law. For a choice of $t = \frac{c_1}{\delta}$, we have a spectral gap $\tilde{\delta} = c_2\delta$ for $\tilde{H}$ as noted before. For some constant c_3, multiplying B and Δ results in

$$\begin{aligned} B\Delta &\leq 4e^{-4l\sqrt{\frac{\tilde{\delta}}{\lambda^\star}}}(2l)^{c_3\frac{l}{u}} \\ &\leq 4e^{-4l\sqrt{\frac{\tilde{\delta}}{\lambda^\star}}+c_3\frac{l}{u}\ln 2l}. \end{aligned}$$

To achieve $B\Delta \leq \frac{1}{2}$, the following must hold

$$\begin{aligned} -\ln 8 &\geq -4l\sqrt{\frac{\tilde{\delta}}{\lambda^\star}} + c_3\frac{l}{u}\ln 2l \\ &= -4l\sqrt{\frac{c_2\delta}{2t+u}} + c_3\frac{l}{u}\ln 2l \\ &= -4l\sqrt{\frac{c_2\delta}{2\frac{c_1}{\delta}+u}} + c_3\frac{l}{u}\ln 2l. \end{aligned}$$

Choosing $u = \sqrt{l}$, we have

$$-4l\sqrt{\frac{c_2\delta}{2\frac{c_1}{\delta}+\sqrt{l}}} + c_3\sqrt{l}\ln 2l \leq -\ln 8.$$

Note that the term $l\sqrt{\frac{c_2\delta}{2\frac{c_1}{\delta}+\sqrt{l}}}$ is asymptotically bigger than $c_3\sqrt{l}\ln 2l$. Choosing $l = c_4/\delta$ for some large enough constant c_4 satisfies this inequality.

Knowing that we are going in the right direction, we proceed to prove the lemma.

Proof. As noted before, we need a clever way to upper bound the Schmidt Rank. Let $(i^\star, i^\star+1)$ be the cut of the AGSP. The polynomial $P_l(\tilde{H})$ is a linear combination of $l+1$ powers of $\tilde{H}$.

We group terms in the expansion of $\tilde{H}^l$ by the number of occurrences of each local Hamiltonian H_j, for $j \in \{L, i-u/2, \ldots, i+u/2-1, R\}$. For a multi-index $\vec{a} \in \{L, i-u/2, \ldots, i+u/2-1, R\}^l$ let $f_{\vec{a}}$ be the sum of all those products of l terms H_j that have a_L copies of H_L, $a_{i-u/2}$ copies of $H_{i-u/2}$, etc. (in any order). An important observation is that for any $\vec{a}$ there is a $j \in \{i-u/2, \ldots, i+u/2-1\}$ such that $j \leq l/u$; this follows from the pigeonhole principle. Hence we can write $\tilde{H}^l = \sum_{j=i-u/2}^{i+u/2-1}\sum_{k=1}^{l/u} Q_{j,k}$, where $Q_{j,k}$ is a sum of a subset of those $f_{\vec{a}}$ for which $a_j = k$. Note, however, that $Q_{j,k}$ cannot always contain *all* the $f_{\vec{a}}$ for which $a_j = k$, as this would be over-counting (e.g. it could be that both $a_i = k$ and $a_{i+1} = k$, in which case we need to choose which of $Q_{i,k}$ and $Q_{i+1,k}$ should contain $f_{\vec{a}}$; we make the decision arbitrarily).

We bound the entanglement rank of each $Q_{j,k}$, across the $(i^\star, i^\star+1)$ cut, individually. For this we first bound the entanglement across the $(j, j+1)$ cut, and make the second important observation that the Schmidt rank across cut i is at most that across cut j times $2^{|i-j|}$: the Schmidt rank cannot increase by more than a factor of 2 for each qubit that we translate the cut by one qubit (exercise!).

Let Z_s be formal commuting variables, and define

$$P_{j,k}(Z) = \sum_{\vec{a}=(a_L, a_{i-u/2}, \ldots, a_{i+u/2-1}, a_R)} f_{\vec{a}}\, Z_L^{a_L} Z_{i-u/2}^{a_{i-u/2}} \cdots Z_{i+u/2-1}^{a_{i+u/2-1}} Z_R^{a_R},$$

where the sum ranges over all tuples such that $a_j = k$ and the remaining a_s sum to $l-k$. By definition each $P_{j,k}(z)$, for any $z \in \mathbb{C}^{u+2}$, lies in the span of the $f_{\vec{a}}$; moreover one can show that the set of all $P_{j,k}(z)$, for z ranging over $\mathbb{C}^{u+2}$, spans the whole linear span (over $\mathbb{C}$) of the $f_{\vec{a}}$ (Exercise! Hint: use induction on the number of variables Z). Since there are at most $t = \binom{l-k+u}{u}$ vectors $\vec{a}$ that space has dimension at most t. It is clear from their definition that the $Q_{j,k}$ lie in the same space: each $Q_{j,k}$ is a linear combination of at most t $P_{j,k}(z)$ (for different values of z).

To conclude it only remains to bound the Schmidt rank of $P_{j,k}(z)$ for arbitrary z. Recall that $P_{j,k}(Z)$ is a formal polynomial collecting all those terms involving exactly k occurrences of H_j. Let A be the product of the $Z_s H_s$ for $s < j$ and B the product of the $Z_s H_s$ for $s > j$. Then A, B commute and hence the terms in $P_{j,k}(Z)$ can be grouped into terms of the form $A^{a_0} B^{b_0} H_i A^{a_1} B^{a_1} H_i \cdots H_i A^{a_k} B^{b_k}$. There are at most $\binom{l+k}{2k+1}$ such terms, each of which has Schmidt rank at most 2^{2k}. Thus: $SR_{(j,j+1)}(P_{j,k}(z)) \le \binom{l+k}{2k+1} 2^{2k}$.

Combining all the elements and using the fact that $k \le \frac{l}{u}$ and $u \le \sqrt{l}$, we are ready to bound $SR(Q_{j,k})$ across the $(i, i+1)$ cut as

$$SR(Q_{i,k}) \le \binom{l-k+u}{u} \binom{l+k}{2k+1} 2^{2k+u} \le l^{O(u)} l^{O(\frac{l}{u})} 2^{2\frac{l}{u}+u} \le (2l)^{O(\frac{l}{u})}.$$

The summation of $\tilde{H}^l = \sum_{j=i-u/2}^{i+u/2-1} \sum_{k=1}^{\frac{l}{u}} Q_{j,k}$ over j and k can be absorbed in the asymptotic notation used in the exponent.

The bound on Δ follows from the previous claim about Chebyshev polynomials. $\square$

References

[1] Arad, I., Kitaev, A., Landau, Z., and Vazirani, U. An area law and sub-exponential algorithm for 1D systems. *CoRR, 1301.1162*, 2013.

[2] Arad, I., Landau, Z., and Vazirani, U. An improved 1D area law for frustration-free systems. *CoRR, 1111.2970*, 2011.