CS286.2 Lecture 10: Detectability Lemma, Decay of Correlations, and Area Law

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In this lecture we will discuss the detectability lemma, decay of correlations, and the area law. Throughout this lecture we will assume that $H = \sum_{i=1}^{m} H_i$ is a 2-local Hamiltonian on qubits such that each $0 \leq H_i \leq 1$ is a projection ($H_i^2 = H_i$) and H is frustration free, meaning that $\lambda_0(H) = 0$ (the lowest eigenvalue is zero) and there is a unique ground state $|\psi_0\rangle$. These conditions are not fully necessary, but they will make out proofs easier. The most significant one is the assumption that $\lambda_0(H) = 0$; everything we say today remains true without it but some proofs become significantly more difficult. In contrast allowing general $0 \leq H_i \leq 1$ and relaxing to q-local on qudits (dimension d particles) is fairly straightforward.

First, recall from last time the detectability lemma, briefly restated here:

Lemma 1 (Detectability Lemma). *Let $\delta = \lambda_1(H) > 0$ be the second lowest eigenvalue of H (the difference $\lambda_1 - \lambda_0$ is known as the spectral gap of the Hamiltonian), and consider the operator $A = (\mathbb{I} - H_m) \ldots (\mathbb{I} - H_1)$. Let $|\psi\rangle$ be any state such that $\langle\psi|\psi_0\rangle = 0$. Then*

$$\|A|\psi\rangle\|^2 \leq 1 - \Omega(\delta),$$

where here Ω hides a dependency on the maximum degree of the interaction graph.

Consider the following quantum example [1]:

Example 2. *Suppose all of the local H_i commute. Let $|\psi\rangle$ be a simultaneous eigenvector with eigenvalue $\lambda(H) = \delta$. We can write*

$$\langle\psi|H|\psi\rangle = \sum_{i=1}^{m} \langle\psi|H_i|\psi\rangle = \sum_{i=1}^{m} a_i$$

where a_i is the penalty for violating each Hamiltonian term. In this case,

$$\|A|\psi\rangle\|^2 = ((1 - a_m) \ldots (1 - a_1))^2 \leq 1 - (\sum_{i=1}^{m} a_i)$$

so we see that in the quantum case, the bound is pretty tight. The $\Omega(\delta)$ dependence comes in when the terms in the Hamiltonian do not commute.

A consequence of the detectability lemma is decay of correlations, which we will now discuss.

Theorem 3 (Decay of Correlations). *Suppose H is a geometrically-local Hamiltonian on a n-qubit D-dimensional grid with spectral gap $\delta > 0$. Let X, Y be Hermitian operators on $(\mathbb{C}^2)^{\otimes n}$ such that $d(X, Y) \geq m$ (here $d(X, Y)$ is the length of the shortest path on the grid between the patches X and Y act on), then,*

$$\left|\langle\psi_0|X|\psi_0\rangle\langle\psi_0|Y|\psi_0\rangle - \langle\psi_0|X \otimes Y|\psi_0\rangle\right| \leq \|X\|\|Y\| \cdot e^{-\Omega(m\cdot\delta/D^3)} \tag{1}$$

[1] Note that in the classical case, this bound is not tight, as we obtain $\|A|\psi\rangle\|^2 = 0$, but the statement is still reasonable

In words, this theorem says that the further apart the operators X and Y are, the less correlated the results will be when we measure the ground state. Here is an example where such decay does *not* happen:

Example 4. *Consider the CAT state* $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\dots0\rangle + |1\dots1\rangle)$ *and let* X *be the Pauli* Z *operator on the ith quit, and* Y *be the Pauli* Z *operator on the jth qubit. Since our state is already expressed in the computation basis, we can easily see that* $\langle\psi|X|\psi\rangle = -1\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 0$ *and likewise* $\langle\psi|Y|\psi\rangle = 0$*. This corresponds to acting with* X *and* Y *independently. However, this state is highly correlated for these operators, as, acting on* $|\psi\rangle$ *first with* X *projects* $|\psi\rangle$ *into the state either* $|0\dots0\rangle$ *or* $|1\dots1\rangle$*, where the outcome of the* Y *measurement is fully determined (the outcomes are always perfectly correlated). Indeed, we can compute* $\langle\psi|X\otimes Y|\psi\rangle = 1$*. So te difference* $|\langle\psi|X|\psi\rangle\langle\psi|Y|\psi\rangle - \langle\psi|X\otimes Y|\psi\rangle| = 1$ *and does not decay exponentially: this state cannot be the ground state of a local, gapped Hamiltonian!*

Now let's prove Theorem 3. We will use the detectability lemma, for a well-chosen ordering of the H_i terms.

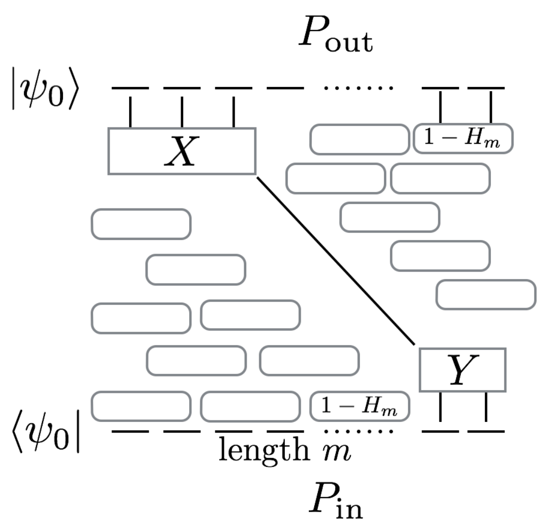

Figure 1: Visual for decay of correlations proof

Proof of Theorem 3. The proof is best described via a picture, expressed in Figure 1. Consider a tensor network representation of $|\psi_0\rangle$with X and Y acting on sets of qubits at distance m. This is represented here in one dimension, but works in any number of dimensions. The picture, without the $1 - H_i$ terms inserted in the middle, corresponds to a tensor nertwork representation of $\langle\psi_0|X \otimes Y|\psi_0\rangle$. Let's consider inserting $1 - H_i$ terms (projectors). Any term which is not in the "causal" cone of X *and* Y (as pictured) can be "absorbed" into the top or bottom states $|\psi_0\rangle$, using $(1 - H_i)|\psi_0\rangle = |\psi_0\rangle$ for any i since $|\psi_0\rangle$ is the ground state and the Hamiltonian is frustration-free. A layer of corresponds to the product of pairwise commuting terms. Inserting all the Hamiltonians requires 2 layers in 1 dimension, and $2D$ layers in dimension D ($2D$ is just the maximum degree). How many layers can we insert? The spacing m between the operators X and Y. Hence overall we can insert $l = m/2D$ copies of the operator $A = \prod_i^m(\mathbb{I} - H_i)$ from the detectability lemma, without changing the value of the tensor network:

$$\langle\psi_0|X \otimes Y|\psi_0\rangle = \langle\psi_0|(X \otimes \mathbb{I})A^l(Y \otimes \mathbb{I})|\psi_0\rangle$$

The detectability lemma implies that

$$\|A^l - |\psi_0\rangle\langle\psi_0|\| \leq (1 - \Omega(\delta/D^2))^l \approx e^{-\Omega(\delta l/D^2)}.$$

Recall that we want to bound

$$\begin{aligned}
&|\langle\psi_0|X|\psi_0\rangle\langle\psi_0|Y|\psi_0\rangle - \langle\psi_0|X \otimes Y|\psi_0\rangle| \\
=&|\langle\psi_0|(X \otimes \mathbb{I})(A^{\frac{m}{2D}} - |\psi_0\rangle\langle\psi_0|)(\mathbb{I} \otimes Y)|\psi_0\rangle| \\
\leq&\|X\| \cdot \|Y\| \cdot \|A^{\frac{m}{2D}} - |\psi_0\rangle\langle\psi_0|\| \leq |X\|\|Y\|e^{-\Omega(m\delta/D^3)},
\end{aligned}$$

which completes the proof. □

Before we discuss the area law, which will essentially tell us that ground states of gapped local Hamiltonians have much less entanglement than the maximum allowed in "generic" states, let's introduce entropy, specifically Von Neumann entropy, as a measure of entanglement.

Definition 5 (von Neumann Entropy)**.** *Consider any $|\psi\rangle \in H_A \otimes H_B$, and perform a Schmidt decomposition to obtain $|\psi\rangle = \sum_{i=1}^d \lambda_i |u_i\rangle_A |v_i\rangle_B$. The von Neumann entropy of $|\psi\rangle$ across region A is defined as*

$$S(A)_{|\psi\rangle} = S\big((\lambda_i^2)_i\big) = \sum_i \lambda_i^2 \ln \frac{1}{\lambda_i^2}. \tag{2}$$

Example 6. *If $|\psi\rangle = |L\rangle \otimes |R\rangle$ then $S(A)_{|\psi\rangle} = 0$ (the product state has no entanglement).*

The maximally entangled state $|\psi\rangle = \sum_i^D \frac{1}{\sqrt{D}}|i\rangle_A \otimes |i\rangle_B$, has entropy $S(A)_{|\psi\rangle} = \ln(D)$. In general, $S(A)_{|\psi\rangle} \leq \ln(\dim \textit{Hilbert space})$. This is a direct quantum generalization of the fact that a classical probability distribution on D elements has entropy at most $\ln D$, and this is achieved by the uniform distribution.

Using this definition, we can now discuss the Area law, stated as a conjecture.

Conjecture 7 (Area Law)**.** *Let $|\psi_0\rangle$ be the ground state of H (the type of Hamiltonian discussed at the beginning of these notes). Then, for any region A,*

$$S(A)_{|\psi_0\rangle} \leq \ln\big(\dim \partial A\big), \tag{3}$$

where ∂A denotes the boundary *of region A, the set of qubits that interact (through H) with at least one qubit outside of region A.*

In other words, the area law conjectures that, in gound states of gapped local Hamiltonians, entropy scales are the surface area, rather than the maximum, which would be the volume. This has recently been proven for 1-dimensions, and we will see a complete proof in this and the next lecture. The conjecture is open for any dimension at least 2.

The proof makes essential use of the following ingredient called Approximate Ground-State Projection (AGSP):

Definition 8 (AGSP)**.** *Given a Hamiltonian H, a (B,Δ)-AGSP for H is an operator $K \in (\mathbb{C}^2)^{\otimes N}$ such that*

- $K|\psi_0\rangle = |\psi_0\rangle$
- $\forall|\psi\rangle : \langle\psi|\psi_0\rangle = 0,\ \|K|\psi\rangle\|^2 \leq \Delta \||\psi\rangle\|^2$
- *If $|\psi\rangle$ has a tensor-network representation with bond dimension $\leq L$, then $K|\psi\rangle$ has a tensor-network representation with bond dimension $\leq B \cdot L$.*

Example 9. *For example, the operator $A = (\mathbb{I} - H_m)\dots(\mathbb{I} - H_1)$ from the detectability lemma is an AGSP. The detectaibility lemma states that its Δ parameter is $\Delta = 1 - \Omega(\delta)$. To evaluate B, observe that applying any $(\mathbb{I} - H_i)$ can be done at the cost of a multiplicative blow-up by a factor at most 4 (the dimension of the space on which $(\mathbb{I} - H_i)$ acts non-trivially) across the bond on which H_i acts. Since each bond is acted on by only one operator, the total blow-up is $B = 4$.*

We will prove a theorem which is a bit more general than the area law, namely

Theorem 10. *Suppose there exists a (B,Δ)-AGSP such that $B\Delta \leq \frac{1}{2}$. Then $|\psi_0\rangle$ satisfies an area law of the form $S(A)_{|\psi_0\rangle} \leq O(1)\log B$.*

First we prove the theorem, based on the following two lemmas. Then we will see a construction of the required (B,Δ)-AGSP. Indeed, note that the AGSP from the detectability lemma does not quite make the cut, since $B\Delta \approx 4$ for small constant δ.

Lemma 11. *Suppose $\exists(B,\Delta)$- AGSP such that $B\Delta \leq \frac{1}{2}$. Fix a partition $(A,\bar{A})$ of the space on which the Hamiltonian acts. Then there exists a product state $|\phi\rangle = |L\rangle_A \otimes |R\rangle_{\bar{A}}$ such that $|\langle\phi|\psi_0\rangle| = \mu \geq \frac{1}{\sqrt{2B}}$.*

Proof. Let $|\phi\rangle$ be a product state with the largest overlap on $|\psi_0\rangle$, meaning that is maximizes $\mu = |\langle\phi|\psi_0\rangle|$, and can be expressed as $|\phi\rangle = \mu|\psi_0\rangle + \sqrt{1-\mu^2}|\psi^\perp\rangle$ (where the latter is some state orthogonal to the ground state). Apply K to get $K|\phi\rangle = \mu|\psi_0\rangle + \delta|\psi'\rangle$ where $|\psi'\rangle$ is normalized and $|\delta|^2 \leq \Delta$. The Schmidt decomposition of $K|\psi_0\rangle$ has at most B terms, so we can express, using Cauchy-Schwarz,

$$\mu = |\langle\psi_0|K\phi\rangle| = |\sum_i \lambda_i \langle\psi_0|L\rangle_i|R\rangle_i| \leq \sqrt{\sum \lambda_i^2}\sqrt{\sum \langle\psi_0|L\rangle_i|R\rangle_i^2} \leq \sqrt{(\mu^2+\Delta)}\sqrt{D}\sqrt{\max_i \langle\psi_0|L\rangle_i|R\rangle_i}$$

Thus there exists a product state such that $|\langle\psi_0|L\rangle_i|R\rangle_i| \geq \dfrac{\mu}{\sqrt{D(\mu^2+\Delta)}}$. This must be $\leq \mu$ by assumption, and hence $\sqrt{D}\sqrt{\mu^2+\Delta} \geq 1$ meaning that $\mu^2 \geq \dfrac{1}{D} - \Delta \geq \dfrac{1}{D} - \dfrac{1}{2D} = \dfrac{1}{2D}$. □

We state the second lemma, and prove it in the next lecture.

Lemma 12. *Suppose $\exists(B,\Delta)$- AGSP such that $B\Delta \leq \frac{1}{2}$ and $\exists$ a product state $|\phi\rangle = |L\rangle_A \otimes |R\rangle_{\bar{A}}$ such that $|\langle\phi|\psi_0\rangle| = \mu$. Then*

$$S(A)_{|\psi_0\rangle} \leq O(1)\frac{\log\mu}{\log\Delta}\log B \tag{4}$$