CS286.2 Around the Quantum PCP Conjecture

Problem sheet 2

9/30/2014

Error reduction for QMA**.** Let V be a QMA verifier whose maximum acceptance probability on n-qubit witnesses $|\xi\rangle$ is p. For any integer k and $1 \leq t \leq k$ let $V_{t,k}$ be the verifier which expects a (nk)-qubit witness $|\xi'\rangle$, processes each chunk of n qubits of $|\xi'\rangle$ independently according to V, and accepts if and only if at least t of the k chunks resulted in the "accept" outcome.

1. Show that the maximum acceptance probability of $V_{k,k}$ is exactly p^k.
2. What is the maximum acceptance probability of $V_{t,k}$, for $1 \leq t \leq k$?
3. Deduce $\mathrm{QMA}_{c,s} \subseteq \mathrm{QMA}_{2/3,1/3}$ for any $c - s = \Omega(1/\operatorname{poly})$. In terms of c and s, how much larger a proof does the modified verifier require?

Upper bounds on QMA**.**

1. Show that QMA $\subseteq$ PSPACE: any language in QMA can be decided by a classical deterministic algorithm running in polynomial space (but possibly exponential time).
2. Improve your upper bound to QMA $\subseteq$ PP, the class of languages decidable in probabilistic polynomial time with unbounded error (inputs in the language should be accepted with probability $> 1/2$, and inputs not in the language with probability $< 1/2$, but there need not be a separation).

A QMA **verification procedure for the local Hamiltonian.** In this exercise we show that $LH_{a,b} \in \mathrm{QMA}$ for any $b - a > 1/\operatorname{poly}(n)$, where n is the number of qubits of the local Hamiltonian.

1. Let $X \in \mathrm{L}(\mathbb{C}^d)$ with $0 \leq X \leq \mathrm{Id}$ be given explicitly, as a $d \times d$ matrix with complex entries. Show how a two-outcome measurement with probabilities $\{\|X^{1/2}|\xi\rangle\|^2, 1 - \|X^{1/2}|\xi\rangle\|^2\}$ can be implemented on state $|\xi\rangle$ by adding a single ancilla qubit initialized in $|0\rangle$, performing a unitary (that you need to specify) on $|\xi\rangle|0\rangle$, and measuring the ancilla in the computational basis.
2. Suppose $H = \sum_{i=1}^m H_i$ given explicitly, and a quantum state $|\xi\rangle$ accessible on a quantum register. Devise a simple randomized procedure that estimates a guess λ for $\langle\xi|H|\xi\rangle$ given access to a single copy of $|\xi\rangle$. What is $E[\lambda]$? Give a simple upper bound on $E[\lambda^2]$. How many copies of $|\xi\rangle$ are required to produce a guess for $\frac{1}{m}\langle\xi|H|\xi\rangle$ that is accurate to within $\pm\varepsilon$?
3. Deduce $LH_{a,b} \in \mathrm{QMA}_{c,s}$ for some c, s depending on a, b. Use error amplification for QMA to conclude $LH_{a,b} \in \mathrm{QMA}_{2/3,1/3} = \mathrm{QMA}$.

5-local Hamiltonian is QMA**-complete.** In class we saw that the the local Hamiltonian problem with Hamiltonians acting on $O(\log n)$ qubits is QMA-hard. In this problem we improve the construction to constant locality, by showing that 5-local Hamiltonian is also QMA-hard. In order to achieve this the main idea is to represent the clock in *unary*: time $|t\rangle$ is represented as $|0\cdots 01\cdots 1\rangle$, with $(T-t)$ zeroes and t ones.

1. Show how to modify H_{in}, H_{out} and H_{prop} so that each term is 5-local and is designed to act on the new clock.

2. The new construction is insufficient: check that invalid clock states, such as $|101\rangle$, have eigenvalue 0 with respect to $H_{in}+H_{out}+H_{prop}$. Design a "penalty" Hamiltonian H_{stab} such that any state of the form $|\psi\rangle|\text{valid clock state}\rangle$ has eigenvalye 0 with respect to H_{stab}, but any state of the form $|\psi\rangle|\text{invalid clock state}\rangle$ has eigenvalue at least 1. What is the locality of H_{stab}?

3. Use the projection lemma to analyze the new Hamiltonian $H = J_{in}H_{in} + H_{out} + J_{prop}H_{prop} + J_{stab}H_{stab}$ and show that it has a small eigenvalue if and only if the original circuit had an accepting state (provided the weights J_{in}, J_{prop} and J_{stab} are chosen appropriately). Conclude.

The Motzkin Hamiltonian. Let $|\Phi\rangle = \frac{1}{\sqrt{2}}(|()\rangle - |00\rangle)$, $|\Psi_l\rangle = \frac{1}{\sqrt{2}}(|(0\rangle - |0(\rangle)$ and $|\Psi_r\rangle = \frac{1}{\sqrt{2}}(|0)\rangle - |)0\rangle)$. The Motzkin Hamiltonian H is

$$H = |)\rangle\langle)|_1 + |(\rangle\langle(|_n + \sum_{j=1}^{n-1}\big(|\Phi\rangle\langle\Phi|_{j,j+1} + |\Psi_l\rangle\langle\Psi_l|_{j,j+1} + |\Psi_r\rangle\langle\Psi_r|_{j,j+1}\big).$$

Let the Motzkin state $|\mathcal{M}_n\rangle$ be the uniform superposition over all well-parenthesized states $|s\rangle$ for $s \in \{(,),0\}$.

1. Let $S_{p,q} \subseteq \{(,),0\}^n$ be the set of all strings s that are equivalent to a string $s_{p,q} =)\cdots)0\cdots0(\cdots($, where there are p left brackets, $(n-(p+q))$ zeros and q right brackets, under local transformations $00 \leftrightarrow ()$, $(0 \leftrightarrow 0($ and $0) \leftrightarrow)0$. Prove that the sets $S_{p,q}$ form a partition of $\{(,),0\}^n$. Show that $S_{0,0}$ is precisely the set of Motzkin paths.

2. Let $\tilde{H} = \sum_{j=1}^{n-1}(|\Phi\rangle\langle\Phi|_{j,j+1} + |\Psi_l\rangle\langle\Psi_l|_{j,j+1} + |\Psi_r\rangle\langle\Psi_r|_{j,j+1})$. Prove that the eigenspace of $\tilde{H}$ associated with eigenvalue 0 (i.e. the ground space) is spanned by the states $|S_{p,q}\rangle$ that are the uniform superposition over all strings in $S_{p,q}$.

3. Conclude that $|\mathcal{M}_n\rangle = |S_{0,0}\rangle$ is the unique ground state of H.