Classification and Computability for Nonlocal Games

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Abstract

Nonlocal games are a convenient reformulation of Bell inequalities in quantum mechanics, and provide a natural setting to investigate advantages provided by access to entanglement. We study two common variants of nonlocal games, namely XOR games and linear system games, which involve players providing (partial) solutions to systems of linear equations. The two types of games have served different purposes: many bounds and structural results have been shown for XOR games, while linear system games have been the setting for undecidability results and have provided separations between classes of quantum correlations. We attempt to relate the two models, by studying the relationships between their strategies and refutations, and try to understand when results for one model can be transferred to the other. We also independently investigate bounds and tractability for XOR games.

Background

Quantum mechanics is now accepted as the basic paradigm for physics on a small scale, mainly due to its ability to explain physical phenomena which previously resisted classical models. However, the superiority of quantum mechanics over classical mechanics as a predictive theory has also been more fully established through experiments designed to test directly for potential violations of classical predictions. The most well-known such experiment was first proposed by John Bell in [Bel64], although it was not experimentally run until years later, and similar *Bell tests* followed. In a typical Bell test, two (or more) non-communicating observers make certain measurements of their respective systems (which are spatially separated but may have previously interacted), according to the instructions of the experimenter. If this experiment is carried out many times, each time with identical preparation, then certain statistical properties emerge among the measured outcomes. In particular, it can be proved that the resulting correlation, i.e. the collection of observed joint probabilites of outcomes given the instructions, must satisfy certain constraints in any theory based on local hidden variables (theories prior to quantum mechanics, e.g. Newtonian mechanics), but that these constraints may be violated in a quantum mechanical world. This provides a concrete separation between the predictions of quantum and classical theories, and in the years since Bell's result, experimental verifications have supported the quantum predictions [BCP$^+$14].

Such constraints as above are known as *Bell inequalities*, and their relationship with the corresponding *Tsirelson bounds* – i.e. the bounds on quantum violations of Bell inequaliites – has been extensively studied. In particular, the difference between these two bounds attests to the power of entanglement as a resource. In this direction, viewing entanglement as a resource, to be potentially used by computers or players in a game, we reformulate Bell tests in the modern language of nonlocal games.

A *nonlocal game* G (with two players, say Alice and Bob), consists of question sets $\mathcal{I}_A$ and $\mathcal{I}_B$ and answer sets $\mathcal{O}_A$ and $\mathcal{O}_B$, and a probability distribution π on $\mathcal{I}_A \times \mathcal{I}_B$, along with a $\{0,1\}$-valued predicate V on $\mathcal{O}_A \times \mathcal{O}_B \times \mathcal{I}_A \times \mathcal{I}_B$ (or more generally an $\mathbb{R}$-valued function on that set). In the game, Alice and Bob are asked questions $x \in \mathcal{I}_A$ and $y \in \mathcal{I}_B$, respectively, with probability $\pi(x, y)$, and they send back respective answers a and b. Their answers follow some conditional probability distribution $p(a, b|x, y)$, and we refer to the set of values $\{p(a, b|x, y)\}$ as a correlation, or a *strategy*. Any proper strategy for the game must be non-signalling, which effectively means that the players cannot communicate, and this condition can be expressed as a finite set of constraints on the values $p(a, b|x, y)$. A classical, or local, strategy is one in which the players' answers only depend on their respective questions, up to some shared randomness λ, and are independent of each other's questions and answers, so we can write

$$p(a, b|x, y) = \int p(a|x, \lambda)p(b|y, \lambda)q(\lambda)\, d\lambda.$$

for some distribution $q(\lambda)$. An entangled strategy is one in which the players have access to entanglement, in that they share an entangled state on which each can perform measurements. We can express this more concretely as follows: an entangled strategy is given by a pair of Hilbert spaces H_A and H_B, one for each player, together with a composite state $|\psi\rangle \in H_A \otimes H_B$, and sets of measurement operators $\{M_a^x \mid a \in \mathcal{O}_A, x \in \mathcal{I}_A\}$ and $\{N_b^y \mid b \in \mathcal{O}_B, y \in \mathcal{I}_B\}$. Here M_a^x is a projection on H_A (or more generally a positive operator) such that $\sum_{a \in A} M_a^x = I_{H_A}$ for each $x \in \mathcal{I}_A$, and the analogous statements hold for the operators N_b^y. The strategy corresponds to the following physical actions in the game: upon receiving questions x, y, respectively, Alice makes the measurement associated to x on H_A, while Bob makes the measurement associated to y on H_B, and they answer according to the outcomes of their measurements. Explicitly, Alice and Bob answer (a, b) with probability $p(a, b|x, y) = \langle\psi|M_a^x \otimes N_b^y|\psi\rangle$. This is the tensor product model of an entangled strategy. More generally, we may allow commuting-operator strategies, in which Alice and Bob share a Hilbert space H and a state $|\psi\rangle \in H$, and, rather than acting on distinct tensor factors, their measurement operators M_a^x and N_b^y (which act on H) are simply required to commute. Similar to before, the correlation is given by $p(a, b|x, y) = \langle\psi|M_a^x N_b^y|\psi\rangle$.

Now, for a given strategy $s = \{p(a, b|x, y)\}$, we define the win probability $w(s)$ to be the probability that $V(a, b|x, y) = 1$ (where x and y are distributed according to π, and for given x and y, a and b are distributed according to $p(a, b|x, y)$). A strategy s is *perfect* if $w(s) = 1$. We define the classical value $\omega(G)$ to be the supremum of $w(s)$ over all classical strategies s, and similarly we define the entangled values $\omega_t^*(G)$ and $\omega_c^*(G)$ to be the suprema of $w(s)$ over all tensor-product and commuting-operator entangled s, respectively. Note in general we have $\omega(G) \leq \omega_t^*(G) \leq \omega_c^*(G)$. Usually we denote $\omega^*(G) = \omega_c^*(G)$. All of the above notions can be generalized to k players (rather than 2) via the same definitions.

Introduction

Let's now introduce the types of games relevant to our research. An XOR game G is usually defined in one of two (effectively equivalent) ways. In the first, a k-XOR game is simply a nonlocal game with k players, such that the answer set for each player is $\{0,1\} = \mathbb{F}_2$, and the predicate V only depends on the sum of the answers mod 2, i.e.

$$V(a_1,\dots,a_k|x_1,\dots,x_k) = V(a_1 \oplus \cdots \oplus a_k|x_1,\dots,x_k).$$

In the other formulation, each player i is assigned variables $x_1^{(i)},\dots,x_n^{(i)}$, and there is a finite system of m equations of the form $x_{j_1}^{(1)} \oplus \cdots \oplus x_{j_k}^{(k)} = b$, such that each clause is chosen with probability $1/m$, and from the chosen clause, each player i is asked to give a value for their associated variable $x_{j_i}^{(i)}$, and the players win if and only if the equation is satisfied by their assignment. We will usually use the second formulation. We can represent a strategy for G as a choice of a (± 1)-valued operator $A_j^{(i)}$ for each player i and associated variable j (together with a shared state $|\psi\rangle$), such that in the tensor product model, $A_j^{(i)}$ acts only on the tensor factor H_i (corresponding to player i) of the composite space $H_1 \otimes \cdots \otimes H_k$, and in the commuting operator model, $A_j^{(i)}$ commutes with $A_{j'}^{(i')}$ whenever $i \neq i'$. Here the projection to the $(+1)$-eigenspace of $A_j^{(i)}$ corresponds to the answer 0, while the projection to the (-1)-eigenspace coresponds to the answer 1. Note that with this, when the players are asked for solutions for the equation $x_{j_1}^{(1)} \oplus \cdots \oplus x_{j_k}^{(k)} = b$, the probability that they answer correctly is

$$\frac{1}{2}\left(1 + (-1)^b \langle\psi|A_{j_1}^{(1)} \cdots A_{j_k}^{(k)}|\psi\rangle\right)$$

In particular, the strategy is perfect if and only if for each equation we have $\langle\psi|A_{j_1}^{(1)} \cdots A_{j_k}^{(k)}|\psi\rangle = (-1)^b$, or equivalently $A_{j_1}^{(1)} \cdots A_{j_k}^{(k)}|\psi\rangle = (-1)^b|\psi\rangle$.

A linear system game is defined somewhat differently. There are several similar formulations, but here is the usual one: here we have a finite set of variables $x_1,\dots,x_n$, and a finite set of m equations of the form $x_{j_1} \oplus \cdots \oplus x_{j_\ell} = b$. Each equation is chosen with some probability (usually $1/m$), and Alice is given the equation, while Bob is given a randomly selected variable from the equation. Alice answers with an assignment of all variables in the equation, while Bob answers with an assignment of his given variable. The players win if Alice's assignment satisfies the equation, and Alice's and Bob's assignments agree at the variable given to Bob.

Both of these types of games ask about solutions of systems of linear equations. This is most direct in the classical case: for a given XOR game or linear system game G, G has a perfect classical strategy, i.e. $\omega = 1$, if and only if there is an assignment of all of the variables in the system which satisfies all equations of the system simultaneously. Thus G has a perfect classical strategy if and only if the linear system has a solution. It is somewhat harder to specify the conditions under which we have $\omega^*(G) = 1$ (in both the XOR and linear system game cases).

In particular, it is known that for a 2-player XOR game, $\omega^* = 1$ if and only if $\omega = 1$ [CHTW04]. This is not the case for XOR games with 3 or more players, as well as for linear system games. For example, the Mermin-Peres Magic Square game is a linear system game with $\omega^* = 1$ but $\omega < 1$ [CM14], while the GHZ game [WHKN18] is a 3-player XOR game with the same property. For linear system games, the condition that $\omega^* = 1$ is characterized by the existence of *quantum satisfying assignments*. For a system of equations of the form $x_{j_1} \oplus \cdots \oplus x_{j_\ell} = b$, define a quantum satisfying assignment to be a Hilbert space H together with a (± 1)-operator A_j on H (i.e. a Hermitian operator with eigenvalues $\lambda = \pm 1$) for each variable x_j, such that A_j and $A_{j'}$ commute whenever x_j and $x_{j'}$ appear in the same equation, and such that for each equation of the above form, we have $A_{j_1} \cdots A_{j_\ell} = (-1)^b I$. Such an assignment is said to be finite-dimensional if H is finite-dimensional. It was shown in [CM14] that $\omega_t^*(G) = 1$ if and only if the system of G has a finite-dimensional quantum satisfying assignment, and it was shown in [CLS17] that $\omega_c^*(G) = 1$ if and only if the system has *any* quantum satisfying assignment. There is also an equivalent formulation in [CLS17] based on solution groups. For G an XOR game, an equivalent condition to $\omega^* = 1$ was found in [WHKN18], via the hierarchy of semidefinite programs from [NPA08]. The condition depends on the notion of a refutation of an XOR game.

The situation is also divided for the computation and approximation of ω^*, as well as for the hardness/tractibility of determining whether $\omega^* = 1$. It was shown in [Slo16], [Slo17] that the problem of determining whether $\omega^* = 1$ for a given linear system game is undecidable. No analogous results are known for XOR games, though it was shown in [Vid13] that it is NP-hard to approximate ω^* for 3-XOR games. From the results in [WHKN18], there is a polynomial-time algorithm to determine if $\omega^* = 1$ for *symmetric* XOR games.

Because of the disparity between these results, despite the alignment in the classical case, it is natural to ask whether we can relate the properties of a linear system, considered as an XOR game G, to the same system considered as a linear system game G'. More broadly, it would be clarifying to find some relationship between XOR games and linear system games which preserves some quality, e.g. a mapping from XOR games to linear system games (or in the other direction) which preserves the property that $\omega^* = 1$, or which simply increases the value of ω^*. Such mappings are desirable because they could potentially allow results which apply to one type of game to be extended to apply to the other. For example, given that determining whether $\omega^* = 1$ is undecidable for linear system games, an appropriate mapping from linear system games to XOR games (i.e. a reduction) could prove that the same problem is undecidable for XOR games.

We will show that the direct reduction from XOR games to linear system games does not preserve any of these properties. Given this result, it remains possible that a less direct reduction exists in either direction, but given the differences between the models, it seems difficult to find one. In lieu of a reduction, we conjecture separately that deciding whether $\omega^* = 1$ for XOR games is undecidable, and study a combinatorial reformulation of this question, giving preliminary results that seem to provide evidence for the conjecture.

Results

At first it seemed that the direct reduction, by simply considering the linear system game G' on the system of equations of an XOR game G, would satisfy the above desired properties. In particular, we had "proofs" of these properties relating $\omega^*(G)$ and $\omega^*(G')$, and proof approaches for other properties. For example, a "proof" that $\omega^*(G) \leq \omega^*(G')$: consider a strategy for G given by a set of operators $\{A_j^{(i)}\}$, where $A_j^{(i)}$ is the measurement corresponding to the j-th variable of player i. We can then construct a corresponding strategy for G': when she receives the equation $x_{j_1}^{(1)} \oplus \cdots \oplus x_{j_k}^{(k)} = b$, Alice measures $A_{j_i}^{(i)}$ for each i (which is possible because they commute) and responds with the sequence of measured values, while when Bob receives the variable $x_j^{(i)}$, he measures $A_j^{(i)}$ and responds with the result. This strategy seems to succeed with the same probability as the original strategy for G, since Alice's and Bob's measurements will always agree at Bob's variable, so they win precisely when Alice's measurements give a satisfying assignment. Unfortunately, this is not in general a proper entangled strategy, since Alice's and Bob's measurements must commute, i.e. all $A_j^{(i)}$ must commute, meaning the original strategy is in fact classical (so we have only proved the obvious $\omega(G) \leq \omega^*(G')$).

Similarly, we seemed to have a proof that if $\omega^*(G') = 1$, then $\omega^*(G) = 1$. The idea was that if $\omega^*(G') = 1$, then by the result in [CLS17], there must be some quantum satisfying assignment $\{A_j^{(i)}\}$ of the system, such that $A_{j_1}^{(1)} \cdots A_{j_k}^{(k)} = (-1)^b$ for each equation $x_{j_1}^{(1)} \oplus \cdots \oplus x_{j_k}^{(k)} = b$. From here, taking $|\psi\rangle$ to be any element of the associated Hilbert space, we would have $A_{j_1}^{(1)} \cdots A_{j_k}^{(k)} |\psi\rangle = (-1)^b |\psi\rangle$ for each equation, hence a perfect strategy for the XOR game G. There is again a problem, in that this is in general not a proper strategy for the XOR game, since we must have that $A_j^{(i)}$ commutes with $A_{j'}^{(i')}$ whenever $i \neq i'$, but this is only guaranteed to be true if $x_j^{(i)}$ and $x_{j'}^{(i')}$ appear together in the same equation.

Assuming the proofs for these two claims had been successful, this would have shown $\omega^*(G) = 1$ if and only if $\omega^*(G') = 1$, which would have been significant, though this is not the desired reduction to prove undecidability of checking $\omega^* = 1$ for XOR games, since the reduction would have to go in the opposite direction, i.e. from linear system games to XOR games. At the time, we also found flawed approaches to prove the stronger claims that $\omega_c^*(G) \leq \omega_t^*(G')$ and $\omega^*(G') \leq \omega^*(G)$, but the faults with these were much easier to identify. It now seems less likely to find a reduction in either direction, since it is unclear how to transfer strategies between the two types of games, because of the distinct types of commutativity requirements imposed by the different formulatinos.

However, we have disproved our earlier claims about the direct reduction from XOR games to linear system games. In particular, we will now construct XOR games G_1 and G_2 such that $\omega^*(G_1) < 1$, yet $\omega^*(G_1') = 1$, and $\omega^*(G_2) = 1$, yet $\omega^*(G_2') < 1$. This shows that there are no inequalities between $\omega^*(G)$ and $\omega^*(G')$, and more generally, that there are no implications between the conditions $\omega^*(G) = 1$ and $\omega^*(G') = 1$. We give the examples below, but first mention some relevant properties of XOR games and linear system games (some of the definitions below are ours).

Let G be an XOR game with m equations of the form $x_{j_1}^{(1)} \oplus \cdots \oplus x_{j_k}^{(k)} = b$. We refer to the left-hand side as a query, and we refer to $(-1)^b$ as the sign of the query. Define the $\mathbb{Z}_2$-free group $F_{\mathbb{Z}_2}(a_1, \ldots, a_r)$ on the generators $a_1, \ldots, a_r$ to be the free group on the same generators, modulo the relations $a_i^2 = 1$ for all i. Then define the (multiplicative) group

$$S(G) = F_{\mathbb{Z}_2}(V_1) \times \cdots \times F_{\mathbb{Z}_2}(V_k) \times \{\pm 1\}$$

where V_i is the set of variables of player i. There is no loss of generality for our purposes in assuming that $V_i = \{1, \ldots, n\}$ for each i, so we have $S(G) = (F_{\mathbb{Z}_2}(1, \ldots, n))^k \times \{\pm 1\}$. We will write elements of $S(G)$ as column vectors, in the form

$$\begin{bmatrix} w_1 \\ \vdots \\ w_k \\ s \end{bmatrix}$$

where each w_i is an element of $F_{\mathbb{Z}_2}(1, \ldots, n)$, i.e. a word in the symbols $1, \ldots, n$ (possibly empty), and $s \in \{\pm 1\}$. In particular, we can associate to each equation $x_{j_1}^{(1)} \oplus \cdots \oplus x_{j_k}^{(k)} = b$ of G an element of $S(G)$, namely

$$\begin{bmatrix} j_1 \\ \vdots \\ j_k \\ (-1)^b \end{bmatrix}$$

and we will call such an element a relation. We define a *refutation* to be an expression of the element

$$J = \begin{bmatrix} \\ \vdots \\ \\ -1 \end{bmatrix}$$

(where rows 1 through k contain the empty word) as a product of relations. It is not hard to show that if G has such a refutation, then it has no perfect strategy, and the authors of [WHKN18] go further, proving that if G has a refutation of length ℓ, then $\omega^*(G) \leq 1 - \frac{\pi^2}{4m\ell^2}$. They also prove an inverse result: if G has no refutation, then $\omega^*(G) = 1$, which follows from an argument using the hierarchy of semidefinite programs found in [NPA08]. Thus $\omega^*(G) = 1$ if and only if G has no refutation. Note that there is a refutation if and only if J is in the subgroup of $S(G)$ generated by the relations of G.

Some parallel results to the above are known for linear system games. Consider a linear system game G', with equations of the form $x_{j_1} \oplus \cdots \oplus x_{j_k} = b$. We can define the solution group $S'(G')$ (as defined in [CLS17]) to be the group with generators x_i (one for each variable of G') and the additional generator J (representing the scalar operator -1) and the following relations: $J^2 = 1$ and $x_i^2 = 1$ for all i, J commutes with all x_i, and x_i and $x_{i'}$ commute whenever they appear in the same equation, and $x_{j_1} \cdots x_{j_k} = J^b$ for each equation $x_{j_1} \oplus \cdots \oplus x_{j_k} = b$. It was shown in [CLS17] that G' has a perfect strategy if and only if $J \neq 1$ in the solution group $S'(G')$. We can reformulate this condition as saying that J is in the normal subgroup generated by the relations. In particular, since the normal subgroup generated by the relations is just the subgroup generated by all conjugates of the relations, we can define a *refutation* for G' as an expression of J as a product of conjugates of the relations. Equivalently, taking all relations except those drawn from the equations of G' to hold (commutativity, squaring to the identity), a refutation is an expression of J as a product of conjugates of elements of the form $x_{i_1} \cdots x_{i_k} J^b$. Thus G' has a perfect strategy if and only if it has no refutations. Since the set of commuting quantum correlations is closed [Fri12], it follows in particular that any game has $\omega^* = 1$ if and only if it has a perfect strategy, so we can rephrase this as $\omega^*(G') = 1$ if and only if G' has no refutations.

Now we have notions of refutations for both the XOR and linear system game formulations, and each reflects the same idea: a combination of the equations of the game which seems to result in a contradiction. The ways they differ reflect the different structures of the two games, in that a refutation for an XOR game assumes that elements corresponding to variables associated to different players commute, while a refutation for a linear system game only assumes that variables which appear in the same equation commute. Moreover, refutations for linear system games allow conjugation, since a perfect strategy for a linear system game necessarily extends to a "global" perfect strategy, i.e. a quantum satisfying assignment, by the result in [CLS17], while refutations for XOR games do not, since such extensions are not always possible for perfect strategies for XOR games. Note that this gives further evidence that we may not be a reduction with the desired properties: the two conditions that $\omega^* = 1$ are significantly different for the two types of groups associated to these games.

Now we define our examples. The 3-XOR game G_1 is based on the Small 123 game found in [WHKN18], and is similar to the Mermin-Peres Magic Square game found in [CHTW04] and [CM14]. Consider the matrix containing the nine variables

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ b_2 & c_2 & a_2 \\ c_3 & a_3 & b_3 \end{pmatrix}.$$

Then G_1 has six equations given by the following constraints: the sum of the variables in each row is 0, while the sum of the variables in each column is 1 (addition is mod 2). Here Alice is associated with the variables a_i, Bob is associated with the variables b_i, and Charlie is associated with the variables c_i. By writing these equations as relations and ordering them appropriately, we see there is a refutation of G_1 given by

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1\\2\\3\\-1\end{bmatrix}\begin{bmatrix}3\\3\\3\\1\end{bmatrix}\begin{bmatrix}2\\3\\1\\-1\end{bmatrix}\begin{bmatrix}2\\2\\2\\1\end{bmatrix}\begin{bmatrix}3\\1\\2\\-1\end{bmatrix}=\begin{bmatrix}\\ \\ \\ -1\end{bmatrix}$$

hence $\omega^*(G_1) < 1$ (and in fact $\omega^*(G_1) \leq 1 - \frac{\pi^2}{864}$). However, we have a finite-dimensional quantum satisfying assignment for this system, inspired by the perfect strategy for the Magic Square game, by setting the operators corresponding to the variables in the above matrix to

$$\begin{pmatrix} X\otimes Y & Y\otimes X & Z\otimes Z \\ Y\otimes Z & Z\otimes Y & X\otimes X \\ Z\otimes X & X\otimes Z & Y\otimes Y \end{pmatrix}$$

where X, Y, Z are the Pauli spin operators (and our Hilbert space is the space on two qubits). Thus $\omega^*(G_1') = 1$.

For the second example, let the 3-XOR game G_2 be the GHZ game as defined in [WHKN18], with two variables per player and four equations, namely

$$\begin{aligned} a_1 \oplus b_1 \oplus c_1 &= 0 \\ a_2 \oplus b_2 \oplus c_1 &= 1 \\ a_2 \oplus b_1 \oplus c_2 &= 1 \\ a_1 \oplus b_2 \oplus c_2 &= 1 \end{aligned}$$

As shown in [WHKN18], this has a perfect XOR strategy, so $\omega^*(G_2) = 1$, but the linear system game G_2 has a refutation, since we have the following conjugated product of relations

$$(a_1b_1c_1)(a_2b_2c_1J)b_2(a_2b_1c_2J)b_2(a_1b_2c_2J)$$

which evaluates to J via the commuting and squaring relations. Thus G_2' has no perfect commuting-operator strategy, hence $\omega^*(G_2') < 1$.

From here we have tried to find other reductions between XOR games and linear system games, but we have focused more on directly proving that results which apply for one type of game also hold for the other. In particular, we have considered the problem (let's call it XOR-REF) of determining whether $\omega^* = 1$ for XOR games. From the combinatorial reformulation of the condition that $\omega^* = 1$ in terms of refutations, it now seems likely that the problem is undecidable. Our approach has mostly involved attempting to find a reduction from a standard undecidable problem. Because of the combinatorial nature of XOR-REF, which involves checking whether a constraint can be satisfied among different rows by concatenating pairs of "tiles", we first tried to reduce from the Post Correspondence Problem, denoted PCP, a well-known undecidable problem frequently used for reductions. The input to PCP is a finite set of "2×1 tiles" $(u_1, v_1), \ldots, (u_n, v_n)$, where each u_i and v_i is a word on a finite alphabet, say $\{a, b\}$, and problem is to determine if we can find a sequence $i_1, \ldots, i_\ell$ with each $i_j \in [n]$ such that $u_{i_1} \cdots u_{i_\ell} = v_{i_1} \cdots v_{i_\ell}$. Equivalently, the problem is to determine if we can concatenate tiles in a way such that the word on the top row is the same as the word on the bottom row.

This problem seems somewhat difficult to reduce from, since it deals strictly with words and does not allow cancellation, in contrast to XOR-REF. A similar, but possibly more appropriate problem, recently proved to be undecidable in [BP09], is the Identity Correspondence Problem (ICP), which takes input of the same form as PCP, except all the u_i and v_i are elements of the free group $F(a, b)$ rather than simply being words on $\{a, b\}$, and the condition for acceptance is instead that there is a sequence $i_1, \ldots, i_\ell$ such that $u_{i_1} \cdots u_{i_\ell} = v_{i_1} \cdots v_{i_\ell} = 1$. This accounts for the cancellation appearing in XOR-REF, but not for the fact that all the letters (words of length 1) have order 2. The latter condition is not so hard to add: note that we can define an injective homomorphism $F(a, b) \to F_{\mathbb{Z}_2}(x, y, z, w)$ into the $\mathbb{Z}_2$-free group on 4 generators via $a \mapsto xy$, $b \mapsto zw$. Then if we define the problem $\mathbb{Z}_2$ICP to be the same as ICP, except where the elements are instead taken from $F_{\mathbb{Z}_2}(x, y, z, w)$, this homomorphism gives a reduction to $\mathbb{Z}_2$ICP, showing that $\mathbb{Z}_2$ICP is also undecidable. We believe that from here, there may exist a technical, gadget-based reduction from $\mathbb{Z}_2$ICP to XOR-REF, but we have not yet found such a reduction.

Conclusions

The results above suggest that there may not be a simple reduction between XOR games and linear system games, and that the two types of games may have fundamentally different properties, their values and strategies reflecting different qualities of a linear system. This is supported by our result that the direct reduction from XOR games to linear system games fails. The same conclusion is also supported by the two respective conditions given for $\omega^* = 1$, which do not seem compatible, as they are associated with different parts of the two associated solution groups. It remains open whether determining if $\omega^* = 1$ is undecidable for XOR games, though the resemblance to other undecidable problems suggests a path to finding a reduction.

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