# Exams

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# Randomness for testing

Randomness used to test a property
(specific state, Bell violation,...)

- Many systems $Y_1 \ldots Y_n$
- Want: testing random subset enough to guarantee global property

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Another property we would like to test:

$$\boxed{\text{Has student } B \text{ learned the data } X_1 \ldots X_n\text{?}}$$

- Want: testing random subset enough to guarantee global learning

# Probabilistic model

Systems:

- Data: $X_1, \ldots, X_n \in \{0,1\}^n$
- Student memory: $B$ (could be classical or quantum)

Modeled by a joint distribution $P_{X_1 \ldots X_n B}$

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Test:

- Exam: $\vec{i} = (\vec{i}_1, \ldots, \vec{i}_k)$ with $\vec{i}_p \in \{1, \ldots, n\}$
- Given $B$ and $\vec{i}$, answer $A^{\vec{i}} = A^{\vec{i}}(B, \vec{i}) \in \{0,1\}^k$
- Grade given by $\mathsf{G}_k = \sum_{p=1}^{k} \mathbf{1}_{X_{\vec{i}_p} = A_p^{\vec{i}}} = k - d_H(X_{\vec{i}}, A^{\vec{i}})$

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Think of the student as the adversary

# Exam strategies

**Notation:** Data: $X_1 \dots X_n \in \{0,1\}^n$ Exam: $\vec{i} = (\vec{i}_1, \dots, \vec{i}_k)$ with $\vec{i}_p \in \{1, \dots, n\}$
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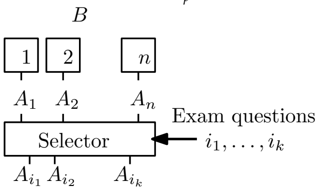
Special kind of strategy:
Simple strategies



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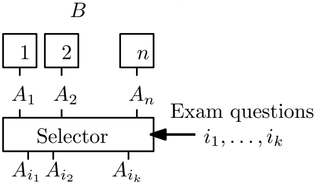

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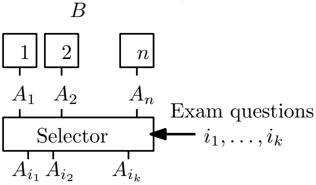
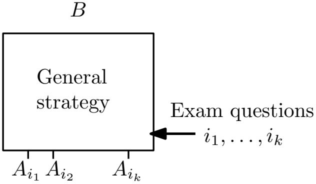



General strategy
described by $\{A^{\vec{i}}\}_{\vec{i}}$




In general strategies, answer to question $i$ depends on context

# Desired statement

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Example: $X$ uniform on $\{0,1\}^n$ and

$$B = \begin{cases} X & \text{with prob. } 1/2 \\ 0 & \text{with prob. } 1/2 \end{cases}$$

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Property we are looking after:

memory $B$ allows answering many $X_1, \ldots, X_n$ correctly

**Theorem**

*For any $P_{X_1 \ldots X_n B}$ and any $\{A^{\vec{i}}\}_{\vec{i}}$, there exists an $\overline{A} = \overline{A}(B, \vec{i}, X_{\vec{i}}) \in \{0,1\}^n$ s.t.*

$$\mathbf{P}\left\{ \frac{\mathsf{G}_n(\overline{A})}{n} \leqslant \frac{\mathsf{G}_k(A^{\vec{i}})}{k} - \delta \right\} \leqslant e^{-\frac{\delta^2 k}{32}}$$

# Exam: simple strategies

**Notation:** Data: $X_1, \ldots, X_n$ Exam: $\vec{i} = (\vec{i}_1, \ldots, \vec{i}_k)$ with $\vec{i}_p \in \{1, \ldots, n\}$

Student memory contains $B$ Answer $A^{\vec{i}} = A^{\vec{i}}(B, \vec{i}) \in \{0,1\}^k$

For simple strategies, $A^{\vec{i}}_p = A_{\vec{i}_p}$ for some $A \in \{0,1\}^n$

We choose $\overline{A} = A$. Statement becomes

**Theorem**

*For any random variable $A = A_1 \ldots A_n$,*

$$\mathbf{P}_{\vec{i},X,A}\left\{\frac{1}{n}\sum_{\ell=1}^{n}\mathbf{1}_{X_\ell = A_\ell} \leqslant \frac{1}{k}\sum_{p=1}^{k}\mathbf{1}_{X_{\vec{i}_p} = A_{\vec{i}_p}} - \delta\right\} \leqslant e^{-\frac{\delta^2 k}{32}}$$

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For fixed $X$ and $A$: standard bounds on hypergeometric distribution

# General case

**Theorem**

*For any strategy* $\{A^{\vec{i}}\}_{\vec{i}}$, *there exists an* $\overline{A} = \overline{A}(B, \vec{i}, X_{\vec{i}})$ *such that*

$$\mathbf{P}\left\{ \frac{\mathsf{G}_n(\overline{A})}{n} \leqslant \frac{\mathsf{G}_k(A^{\vec{i}})}{k} - \delta \right\} \leqslant e^{-\frac{\delta^2 k}{32}}$$

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# Proof: introducing sequential exams

Notation: $\vec{i}^{p-1} = \vec{i}_1, \ldots, \vec{i}_{p-1}$

Two steps:

1. Introduce "sequential exams" and prove statement
2. Relate general strategies for a strategy for sequential exam

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**Sequential exam** (or oral exam)

Interaction between examiner and student

- Questions one by one
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$\rightarrow A_p^{\text{seq}} = A_p^{\text{seq}}(\vec{i}_p, \vec{i}^{p-1}, X_{\vec{i}^{p-1}})$ but independent of $\vec{i}_{p+1} \ldots \vec{i}_k$

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Remark: A priori “general” and “sequential” incomparable

# Proof for sequential strategies

**Theorem**

*For any sequential strategy $A^{\text{seq}}$, there exists an $\overline{A} = \overline{A}(B, \vec{i}, X_{\vec{i}})$ such that*

$$\mathbf{P}\left\{ \frac{\mathsf{G}_n(\overline{A})}{n} \leqslant \frac{\mathsf{G}_k(A^{\vec{i}})}{k} - \delta \right\} \leqslant e^{-\frac{\delta^2 k}{8}}$$

Simplification: define $\overline{A}$ only on $\vec{j}$ with $|\vec{j}| = k$

$\rightarrow$ look at $\left[ \frac{\mathsf{G}_k(\overline{A})}{k} \leqslant \frac{\mathsf{G}_k(A^{\text{seq}})}{k} - \delta \right]$

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$$\overline{A}_{\vec{j}_p} \stackrel{\text{def}}{=} A_p^{\text{seq}}(\vec{j}_p, \vec{i}^{p-1}, X_{\vec{i}^{p-1}})$$

*To answer all questions:* Partition $[n] = S_1 \cup \cdots \cup S_k$ at random, and for $\ell \in S_p$, let $\overline{A}_\ell = A^{\text{seq}}(\ell, \vec{i}^{p-1}, X_{\vec{i}^{p-1}})$

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Using Azuma's inequality:

$$\mathbf{P}\left\{ \sum_{p=1}^{k} \mathbf{1}_{X_{\vec{i}_p} = A_p^{\text{seq}}} - \sum_{p=1}^{k} \mathbf{E}_{\vec{i}_p, X_{\vec{i}_p}} \left\{ \mathbf{1}_{X_{\vec{i}_p} = A_p^{\text{seq}}} \middle| \vec{i}^{p-1} \vec{j}^{p-1} X_{\vec{i}^{p-1}} X_{\vec{j}^{p-1}} \right\} \geqslant \delta k \right\} \leqslant e^{-\frac{\delta^2 k}{2}}$$

$$\mathbf{P}\left\{ \sum_{p=1}^{k} \mathbf{E}_{\vec{j}_p, X_{\vec{j}_p}} \left\{ \mathbf{1}_{X_{\vec{j}_p} = \overline{A}_p} \middle| \vec{i}^{p-1} \vec{j}^{p-1} X_{\vec{i}^{p-1}} X_{\vec{j}^{p-1}} \right\} - \sum_{p=1}^{k} \mathbf{1}_{X_{\vec{j}_p} = \overline{A}_p} \geqslant \delta k \right\} \leqslant e^{-\frac{\delta^2 k}{2}}$$

Observe:

$$\mathbf{E}\left\{ \mathbf{1}_{X_{\vec{i}_p} = A_p^{\text{seq}}} \middle| \vec{i}^{p-1} \vec{j}^{p-1} X_{\vec{i}^{p-1}} X_{\vec{j}^{p-1}} \right\} = \mathbf{E}\left\{ \mathbf{1}_{X_{\vec{j}_p} = \overline{A}_p} \middle| \vec{i}^{p-1} \vec{j}^{p-1} X_{\vec{i}^{p-1}} X_{\vec{j}^{p-1}} \right\}$$

$$\Rightarrow \qquad \mathbf{P}\left\{ \sum_{p=1}^{k} \mathbf{1}_{X_{\vec{i}_p} = A_p^{\text{seq}}} - \sum_{p=1}^{k} \mathbf{1}_{X_{\vec{j}_p} = \overline{A}_p} \geqslant 2\delta k \right\} \leqslant 2e^{-\frac{\delta^2 k}{2}}$$

# Relating general to sequential strategies

**Theorem**

*There exists $A^{\text{seq}}$ such that for any $\{A^{\vec{i}}\}_{\vec{i}}$*

$$\mathbf{P}\left\{\sum_{p=1}^{k}\mathbf{1}_{X_{\vec{i}_p}=A^{\vec{i}}_p}-\sum_{p=1}^{k}\mathbf{1}_{X_{\vec{i}_p}=A^{\text{seq}}_p}\geqslant\delta k\right\}\leqslant e^{-\frac{\delta^2 k}{8}}$$

$$A^{\text{seq}}(\ell,\vec{i}^{p-1},X_{\vec{i}^{p-1}})\stackrel{\text{def}}{=}\text{ best guess for } X_\ell \text{ given } B,\vec{i}^{p-1},X_{\vec{i}^{p-1}}$$

$$\text{Azuma}\quad\Rightarrow\quad\sum_{p=1}^{k}\mathbf{1}_{X_{\vec{i}_p}=A^{\vec{i}}_p}\approx\sum_{p=1}^{k}\mathbf{E}_{\vec{i}_p,X_{\vec{i}_p}}\left\{\mathbf{1}_{X_{\vec{i}_p}=A^{\vec{i}}_p}\middle|\vec{i}^{p-1}X_{\vec{i}^{p-1}}(A^{\vec{i}})^{p-1}\right\}$$

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For any fixed $\vec{i}_p$ and $B$, $(A^{\vec{i}})^{p-1}$ cannot help in predicting $X_{\vec{i}_p}$

$$\Longrightarrow\mathbf{E}_{\vec{i}_p,X_{\vec{i}_p}}\left\{\mathbf{1}_{X_{\vec{i}_p}=A^{\text{seq}}_p}\middle|\vec{i}^{p-1}X_{\vec{i}^{p-1}}(A^{\vec{i}})^{p-1}\right\}\geqslant\mathbf{E}_{\vec{i}_p,X_{\vec{i}_p}}\left\{\mathbf{1}_{X_{\vec{i}_p}=A^{\vec{i}}_p}\middle|\vec{i}^{p-1}X_{\vec{i}^{p-1}}(A^{\vec{i}})^{p-1}\right\}$$

# Recap of the proof

Start with $\{A^{\vec{i}}\}_{\vec{i}}$

$A^{\text{seq}}$ (guesses optimally given past) is at least as good on exam $\vec{i}$

If $A^{\text{seq}}$ works on $\vec{i}$, also works on rest

**Theorem**

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# Quantum student $B$

System $B$ is quantum

Important difficulty: Applying $A^{\vec{i}}(B,\vec{i})$ affects $B$
Issues in the two steps of the classical proof

1. For sequential strategies, measurement may lead to losses
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**Better way of quantifying correlation:**

- Use $H_{\max}(X_1 \dots X_n|B)$ (number of bits of hint for perfect recovery of $X_1 \dots X_n$)