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SPECTRAHEDRAL REGRESSION*

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Abstract. Convex regression is the problem of fitting a convex function to a data set consisting of input-output pairs. We present a new approach to this problem called spectrahedral regression, in which we fit a spectrahedral function to the data, i.e., a function that is the maximum eigenvalue of an affine matrix expression of the input. This method represents a significant generalization of polyhedral (also called max-affine) regression, in which a polyhedral function (a maximum of a fixed number of affine functions) is fit to the data. We prove bounds on how well spectrahedral functions can approximate arbitrary convex functions via statistical risk analysis. We also analyze an alternating minimization algorithm for the nonconvex optimization problem of fitting the best spectrahedral function to a given data set. We show that this algorithm converges geometrically with high probability to a small ball around the optimal parameter given a good initialization. Finally, we demonstrate the utility of our approach with experiments on synthetic data sets as well as real data arising in applications such as economics and engineering design.

Key words. convex regression, support function estimation, semidefinite programming, approximation of convex bodies

MSC codes. 90C22, 90C47, 62G08, 62G20

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1. Introduction.

The problem of identifying a function that approximates a given dataset of input-output pairs is a central one in data science. In this paper we consider the problem of fitting a convex function to such input-output pairs, a task known as *convex regression*. Concretely, given data $\{x^{(i)}, y^{(i)}\}_{i=1}^n \subset \mathbb{R}^d \times \mathbb{R}$, our objective is to identify a convex function $\hat{f}$ such that $\hat{f}(x^{(i)}) \approx y^{(i)}$ for each $i = 1, \ldots, n$. In some applications, one seeks an estimate $\hat{f}$ that is convex and positively homogeneous; in such cases, the problem may equivalently be viewed as one of identifying a convex set given (possibly noisy) support function evaluations. Convex reconstructions in such problems are of interest for several reasons. First, prior domain information in the context of a particular application might naturally lead a practitioner to seek convex approximations. One prominent example arises in economics, in which the theory of marginal utility implies an underlying convexity relationship. Another important example arises in computed tomography applications in which one has access to support function evaluations of some underlying set, and the goal is to reconstruct the set; here, due to the nature of the data acquisition mechanism, the set may be assumed to be convex without loss of generality. A second reason for preferring a

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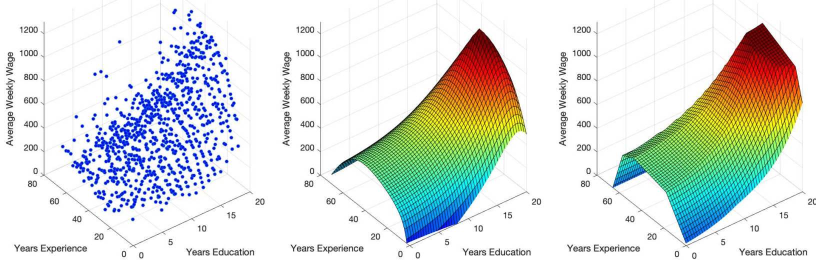

FIG. 1. *Models for average weekly wage based on years of experience and education using spectrahedral and polyhedral regression. From left to right: The underlying data set, the spectrahedral ($m = 3$) estimator, and the polyhedral ($m = 6$) estimator. A transformation in the years of education covariate gives a data set that is approximately convex.*

convex reconstruction $\hat{f}$ is computational—in some applications the goal is to subsequently use $\hat{f}$ as an objective or constraint within an optimization formulation. For example, in aircraft design problems, the precise relationship between various attributes of an aircraft is often not known in closed form, but input-output data are available from simulations; in such cases, identifying a good convex approximation for the input-output relationship is useful for subsequent aircraft design using convex optimization.

A natural first estimator one might write down is

$$\hat{f}^{(n)}_{\mathrm{LSE}} \in \arg\min_{f:\mathbb{R}^d\to\mathbb{R} \text{ is a convex function}} \quad \frac{1}{n}\sum_{i=1}^{n}(y^{(i)} - f(x^{(i)}))^2. \tag{1.1}$$

There always exists a polyhedral function that attains the minimum in (1.1), and this function may be computed efficiently via convex quadratic programming [21, 22, 25]. However, this choice suffers from a number of drawbacks. For a large sample size, the quality of the resulting estimate suffers from overfitting as the complexity of the reconstruction grows with the number of data points. For small sample sizes, the quality of the resulting estimate is often poor due to noise. From a statistical perspective, the estimator may also be suboptimal [16, 17]. For these reasons, it is of interest to regularize the estimator by considering a suitably constrained class of convex functions.

The most popular approach in the literature to penalize the complexity of the reconstruction in (1.1) is to fit a polyhedral function that is representable as the maximum of at most m affine functions (for a user-specified choice of m) to the given data [4, 10, 11, 13, 19, 28], which is based on the observation that convex functions are suprema of affine functions. However, this approach is inherently restrictive in situations in which the underlying phenomenon is better modeled by a nonpolyhedral convex function, which may not be well-approximated by m-polyhedral functions. Further, in settings in which the estimated function is subsequently used within an optimization formulation, the above approach constrains one to using linear-programming (LP) representable functions. See Figure 1 for a demonstration with economic data.

To overcome these limitations, we consider fitting spectrahedral functions to data. To define this model class, let $\mathbb{S}^m_k$ denote the set of $m \times m$ real symmetric matrices that are block-diagonal with blocks of size at most $k \times k$, with k dividing m.

DEFINITION 1.1. *Fix positive integers m,k such that k divides m. A function $f:\mathbb{R}^d\rightarrow\mathbb{R}$ is called (m,k)-spectrahedral if it can be expressed as follows:*

$$f(x)=\lambda_{\max}\left(\sum_{i=1}^d A_i x_i + B\right),$$

where $A_1,\ldots,A_d,B\in\mathbb{S}_k^m$. Here $\lambda_{\max}(\cdot)$ is the largest eigenvalue of a matrix.

An (m,k)-spectrahedral function is convex as it is a composition of a convex function with an affine map. For the case $k=1$, the matrices $A_1,\ldots,A_d,B$ are all diagonal and we recover the case of m-polyhedral functions. The case $k=2$ corresponds to second-order-cone-programming (SOCP) representable functions, and the case $k=m$ utilizes the expressive power of semidefinite programming (SDP). In analogy to the enhanced modeling power of SOCP and SDP in comparison to LP, the class of (m,k)-spectrahedral functions is much richer than the set of m-polyhedral functions for general $k>1$. For instance, when $k=m=d+1$ this class contains the function $f(x)=\|x\|_2$ for $x\in\mathbb{R}^d$ as illustrated in (3.3). For estimates that are (m,k)-spectrahedral, subsequently employing them within optimization formulations yields optimization problems that can be solved via SOCP and SDP.

An (m,k)-spectrahedral function that is positively homogeneous (i.e., $B=0$ in the definition above) is the support function of a convex set that is expressible as the linear image of an (m,k)-spectraplex defined, for positive integers k and m such that k divides m, by

$$\mathcal{S}_{m,k}=\{M\in\mathbb{S}_k^m \mid \operatorname{tr}(M)=1,\ M\succeq 0\}. \tag{1.2}$$

We refer to the collection of linear images of $\mathcal{S}_{m,k}$ as (m,k)-*spectratopes*. Again, the case $k=1$ corresponds to the m-simplex, and the corresponding linear images are m-polytopes. Thus, in the positively homogeneous case, our proposal is to identify a linear image of an (m,k)-spectraplex to fit a given set of support function evaluations. We note that the case $k=m$ was recently considered in [27], and we comment in more detail on the comparison between the present paper and [27] in section 1.2.

1.1. Our contributions. We consider the following constrained analogue of (1.1):

$$\hat{f}_{m,k}^{(n)}\in\arg\min_{f:\mathbb{R}^d\rightarrow\mathbb{R}\text{ is an }(m,k)\text{-spectrahedral function}}\ \frac{1}{n}\sum_{i=1}^n(y^{(i)}-f(x^{(i)}))^2. \tag{1.3}$$

Here the parameters m,k are specified by the user.

First, we investigate in section 2 the expressive power of (m,k)-spectrahedral functions. Our approach to addressing this question is statistical in nature and it proceeds in two steps. We begin by deriving upper bounds on the error of the constrained estimator (1.3) (under suitable assumptions on the data $\{(x^{(i)},y^{(i)})\}_{i=1}^n$ supplied to the estimator (1.3)), which entails computing the pseudo-dimension of a set that captures the complexity of the class of spectrahedral functions. As is standard in statistical learning theory, this error decomposes into an estimation error (due to finite sample size) and an approximation error (due to constraining the estimator (1.3) to a proper subclass of convex functions). We then compare these to known minimax lower bounds on the error of any procedure for identifying a convex function [10, 28]. Combined together, for the case of fixed k (as a function of m) we obtain tight lower bounds on how well an (m,k)-spectrahedral function can approximate a Lipschitz convex function over a compact convex domain, and on how well a linear image of an (m,k)-spectraplex can approximate an arbitrary convex body (see Theorem 2.8). To the best of our knowledge, such bounds have only been obtained previously in the literature for the case $k=1$, e.g., how well m-polytopes can approximate arbitrary convex bodies [3, 6].

Second, we investigate in section 3 the performance of an alternating minimization procedure to solve (1.3) for a user-specified m,k. This method is a natural generalization of a widely used approach for fitting m-polyhedral functions, and it was first described in [27] for the case of positively homogeneous convex regression with $k=m$. We investigate the convergence properties of this algorithm under the following problem setup. Consider an (m,k)-spectrahedral function $f_* : \mathbb{R}^d \to \mathbb{R}$. Assuming that the covariates $x^{(i)},\ i=1,\dots,n$, are independent and identically distributed (i.i.d.) sub-Gaussian and each $y^{(i)} = f_*(x^{(i)}) + \varepsilon_i,\ i=1,\dots,n$, for i.i.d. Gaussian noise ε_i, we show in Theorem 3.1 that the alternating minimization algorithm is locally linearly convergent with high probability given sufficiently large n. A key feature of this analysis is that the requirements on the sample size n and the assumptions on the quality of the initial guess are functions of a "condition number" type quantity associated to f_*, which (roughly speaking) measures how f_* changes if the parameters that describe it are perturbed. The assumption on f_* in Theorem 3.1 may, however, be difficult to satisfy when $k < m$. We show in Theorem 3.2 that a similar convergence guarantee holds under a weaker condition on f_* at the expense of stronger assumptions on the distribution of the covariates.

Finally, in section 4 we give empirical evidence of the utility of our estimator (1.3) on both synthetic datasets as well as data arising from real-world applications.

1.2. Related work. There are three broad topics with which our work has a number of connections, and we describe these in detail next.

First, we consider our results in the context of the recent literature in optimization on lift-and-project methods (see the recent survey [7] and the references therein). This body of work has studied the question of the most compact description of a convex body as a linear image of an affine section of a cone, and has provided lower bounds on the sizes of such descriptions for prominent families of cone programs such as LP, SOCP, and SDP. This literature has primarily considered exact descriptions, and there is relatively little work on lower bounds for approximate descriptions (with the exception of the case of polyhedral descriptions). The present paper may be viewed as an approximation-theoretic complement to this body of work, and we obtain tight lower bounds on the expressive power of (m,k)-spectrahedral functions (and on linear images of the (m,k)-spectraplex) for bounded $k>1$.

Second, recent results provide algorithmic guarantees for the widely used alternating minimization procedure for fitting m-polyhedral functions [8, 9]; this work gives both a local convergence analysis as well as a dimension reduction strategy to restrict the space over which one needs to consider random initializations. In comparison, our results provide only a local convergence analysis, although we do so for a more general alternating minimization procedure that is suitable for fitting general (m,k)-spectrahedral functions. We defer the study of a suitable initialization strategy to future work (see section 5).

Finally, we note that there is prior work on fitting nonpolyhedral functions in the convex regression problem. Specifically, [14] suggests various heuristics to fit a log-sum-exp type function, which may be viewed as a "soft-max" function. However, these methods do not come with any approximation-theoretic or algorithmic guarantees. The recent work [27] considered the problem of fitting a convex body given support function evaluations, i.e., the case of positively homogeneous convex regression, and proposed reconstructions that are linear images of an (m,m)-spectraplex; in this context, [27] provided an asymptotic statistical analysis of the associated estimator and first described an alternating minimization procedure that generalized the m-polyhedral case, but with no algorithmic guarantees. In comparison to [27], the present paper considers the more general setting of convex regression and also allows for the spectrahedral function to have additional block-diagonal structure, i.e., general (m,k)-spectrahedral reconstructions. Further, we provide algorithmic guarantees in the form of local convergence analysis of the alternating minimization procedure, and we provide approximation-theoretic guarantees associated to (m,k)-spectrahedral functions (which rely on finite sample rather than asymptotic statistical analysis).

1.3. Notation. For $\mathcal{A} = (A_1, \ldots, A_d) \in (\mathbb{S}_k^m)^d$, we define for $x \in \mathbb{R}^d$ the linear pencil $\mathcal{A}[x] := \sum_{i=1}^d x_i A_i \in \mathbb{S}_k^m$. The usual vector ℓ_2 norm is denoted $\|\cdot\|_2$ and the sup norm by $\|\cdot\|_\infty$. The matrix Frobenius norm is denoted by $\|\cdot\|_F$, and the matrix operator norm by $\|\cdot\|_{op}$. We denote by $B_d(x,R)$ the ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $R > 0$.

2. Expressiveness of spectrahedral functions via statistical risk bounds.

In this section, we first obtain upper bounds on the risk of the (m,k)-spectrahedral estimator in (1.3) decomposed into the approximation error and estimation error. We then compare this upper bound with known minimax lower bounds on the risk for certain classes of convex functions. This provides lower bounds on the approximation error of (m,k)-spectrahedral functions to these functions classes.

2.1. General upper bound on the risk.

To obtain an upper bound on the risk of the estimator (1.3), we use the general bound obtained in [11, section 4.1]. To give the statement, consider first the following general framework. Let $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$ be observations satisfying

$$y^{(i)} = f_*(x^{(i)}) + \varepsilon_i, \tag{2.1}$$

for a function $f_* : \mathbb{R}^d \to \mathbb{R}$ contained in some function class $\mathcal{F}$. We assume the errors ε_i are i.i.d. mean zero Gaussians with variance σ^2. Now, let $\{\mathcal{F}_m\}_{m \in \mathbb{N}}$ be a collection of function classes of growing complexity with m. For each m, define the constrained least squares estimator

$$\hat{f}_m^{(n)} := \mathrm{argmin}_{f \in \mathcal{F}_m} \sum_{i=1}^n (y^{(i)} - f(x^{(i)}))^2.$$

We consider the risk of this estimator in the random design setting,[1] where we assume $x^{(1)}, \ldots, x^{(n)}$ are i.i.d. random vectors in $\mathbb{R}^d$ with distribution μ. The risk is then defined by

$$\|\hat{f}_m^{(n)} - f_*\|_\mu^2 := \int_{\mathbb{R}^d} (\hat{f}_m^{(n)}(x) - f_*(x))^2 \mathrm{d}\mu(x).$$

[1] One can also consider the risk in the fixed design setting, where one assumes the covariates $\{x^{(i)}\}_{i=1}^n$ are fixed, and risk bounds proved in [11] include this case. The results in this work can be directly extended to this case as well by applying the corresponding results.

Additionally, assume both f_* and $\mathcal{F}_m$ are uniformly bounded by a positive and finite constant Γ.

As is standard in the theory of empirical processes, the rate is determined by the complexity of the class $\mathcal{F}_m$, which in this case is determined by the pseudo-dimension of the set

$$H_m := \{z \in \mathbb{R}^n : z = (f(x^{(1)}), \dots, f(x^{(n)})) \text{ for some } f \in \mathcal{F}_m\}. \tag{2.2}$$

Recall that the pseudo-dimension of subset $B \subset \mathbb{R}^n$, denoted by $\mathrm{Pdim}(B)$, is defined as the maximum cardinality of a subset $\sigma \subseteq \{1, \dots, n\}$ for which there exists $h \in \mathbb{R}^n$ such that for every $\sigma' \subseteq \sigma$, one can find $a \in B$ with $a_i < h_i$ for $i \in \sigma'$ and $a_i > h_i$ for $i \in \sigma \backslash \sigma'$.

Theorem 4.2 in [11], stated below, provides an upper bound on the risk of $\hat{f}_m^{(n)}$ split into approximation error and estimation error.

THEOREM 2.1. *Let $n \geq 7$. Suppose there is a constant $D_m \geq 1$ such that $\mathrm{Pdim}(H_m) \leq D_m$. Then, there exists an absolute constant c such that*

$$\mathbb{E}\left[\|\hat{f}_m^{(n)} - f_*\|_\mu^2\right] \leq c\left(\inf_{f \in \mathcal{F}_m} \|f - f_*\|_\mu^2 + \frac{\max\{\sigma^2, \Gamma^2\} D_m \log n}{n}\right). \tag{2.3}$$

The (m,k)-spectrahedral estimator (1.3) is a special case of the estimator $\hat{f}_m^{(n)}$ when $\mathcal{F}$ is the class of convex functions $f : \mathbb{R}^d \to \mathbb{R}$ and $\mathcal{F}_m$ is the class of (m,k)-spectrahedral functions as in Definition 1.1, denoted by $\mathcal{F}_{m,k}$. Since the class is parameterized by $d+1$ matrices in $\mathbb{S}_k^m$, we define, for each $m \in \mathbb{N}$ and $k = 1, \dots, m$,

$$(\hat{A}_1, \dots, \hat{A}_d, \hat{B}) \in \operatorname{argmin}_{A_1,\dots,A_d,B \in \mathbb{S}_k^m} \sum_{j=1}^n \left[y^{(j)} - \lambda_{\max}\left(\sum_{i=1}^d x_i^{(j)} A_i + B\right)\right]^2, \tag{2.4}$$

and we define the (m,k)-spectrahedral estimator of f_* by

$$\hat{f}_{m,k}(x) := \lambda_{\max}\left(\sum_{i=1}^d x_i \hat{A}_i + \hat{B}\right).$$

We also define the estimator when $\mathcal{F}$ is the class of support functions of convex bodies (compact and convex subsets) in $\mathbb{R}^d$, denoted by $\mathcal{K}$, and $\mathcal{F}_m$ is the subclass consisting of positively homogeneous (m,k)-spectrahedral functions, or equivalently, support functions of (m,k)-spectratopes. This corresponds to the case when the offset matrix $B = 0$. In this setting, we assume we are given observations $(u^{(1)}, y^{(1)}), \dots, (u^{(n)}, y^{(n)}) \in S^{d-1} \times \mathbb{R}$ satisfying

$$y^{(i)} = h_{K_*}(u^{(i)}) + \varepsilon_i,$$

where $h_K(u) := \sup_{x \in K} \langle u, x \rangle$, $u \in S^{d-1}$, is the support function of a set $K_* \in \mathcal{K}$. We denote the class of (m,k)-spectratopes or linear images of $\mathcal{S}_{m,k}$ in $\mathbb{R}^d$ by $\mathcal{L}(\mathcal{S}_{m,k})$. To define the (m,k)-spectratope estimator, let

$$(\hat{A}_1, \dots, \hat{A}_d) \in \operatorname{argmin}_{A_1,\dots,A_d, \in \mathbb{S}_k^m} \sum_{j=1}^n \left[Y_i - \lambda_{\max}\left(\sum_{i=1}^d u_i^{(j)} A_i\right)\right]^2,$$

and define

$$\hat{K}_{m,k} := \{z \in \mathbb{R}^d : z = (\langle \hat{A}_1, X\rangle, \dots, \langle \hat{A}_d, X\rangle) \text{ for some } X \in \mathcal{S}_{m,k}\}.$$

In this support function estimation setting, we notate the risk in terms of the convex bodies. Letting ν denote the probability distribution on S^{d-1} of $u^{(1)}$, we define the risk

$$\ell_\nu^2(\hat{K},K) := \int_{S^{d-1}} (h_{\hat{K}}(u) - h_K(u))^2 \mathrm{d}\nu(u).$$

In the following lemma, we prove an upper bound on the pseudo-dimension of the relevant set (2.2) needed to apply Theorem 2.1 for the estimators $\hat{f}_{m,k}$ and $\hat{K}_{m,k}$.

LEMMA 2.2. *For $m,k \in \mathbb{N}$ such that k divides m, define, for $x^{(1)},\ldots,x^{(n)} \in \mathbb{R}^d$,*

$$H_{m,k} := \Big\{ z = \Big(\lambda_{\max}\Big(\mathcal{A}[x^{(1)}] + B\Big),\ldots,\lambda_{\max}\Big(\mathcal{A}[x^{(n)}] + B\Big)\Big) \in \mathbb{R}^n \text{ for some } \mathcal{A} \in (\mathbb{S}_k^m)^d, B \in \mathbb{S}_k^m \Big\},$$

and for $u^{(1)},\ldots,u^{(d)} \in \mathbb{S}^{d-1}$,

$$\tilde{H}_{m,k} := \Big\{ z = \Big(\lambda_{\max}\Big(\mathcal{A}[u^{(1)}]\Big),\ldots,\lambda_{\max}\Big(\mathcal{A}[u^{(n)}]\Big)\Big) \in \mathbb{R}^n \text{ for some } \mathcal{A} \in (\mathbb{S}_k^m)^d \Big\},$$

Then, there exist absolute constants $c_1,c_2 > 0$ such that

$$Pdim(H_{m,k}) \le c_1 km(d+1)\log(c_2 n/k) \quad \text{and} \quad Pdim(\tilde{H}_{m,k}) \le c_1 kmd\log(c_2 n/k).$$

To prove the lemma, we need the following known result (see for instance, Lemma 2.1 in [1]).

PROPOSITION 2.3. *Let $p_1,\ldots,p_n$ be fixed polynomials of degree at most m in D variables for $D \le m$. The number of distinct sign vectors $(\mathrm{sgn}(p_1(A)),\ldots,\mathrm{sgn}(p_n(A)))$ that can be obtained by varying $A \in \mathbb{R}^D$ is at most $2\left(\frac{2enm}{D}\right)^D$.*

Proof of Lemma 2.2. Assume that the pseudo-dimension of $H_{m,k} \subset \mathbb{R}^n$ is ρ. By the definition of pseudo-dimension, the size of the collection of sign vectors

$$\mathcal{G}_{m,k} := \{(\mathrm{sgn}(\lambda_{\max}(\mathcal{A}[x^{(1)}] + B)),\ldots,\mathrm{sgn}(\lambda_{\max}(\mathcal{A}[x^{(n)}] + B))) : \mathcal{A} \in (\mathbb{S}_k^m)^d, B \in \mathbb{S}_k^m\}$$

must be at most 2^ρ. For each i,

$$\mathrm{sgn}(\mathcal{A}[x^{(i)}] + B) = \mathrm{sgn}(\min\{p_1(\mathcal{A},B;x^{(i)}),\ldots,p_m(\mathcal{A},B;x^{(i)})\}),$$

where $p_\ell(\mathcal{A},B;x^{(i)}) = \det(-(\mathcal{A}[x^{(i)}] + B)_{\ell:\ell})$ is the determinant of the $\ell \times \ell$ principal submatrix of $-\mathcal{A}[x^{(i)}] - B$. Indeed, $\lambda_{\max}(\mathcal{A}[x^{(i)}] + B) \le 0$ if and only if all of these determinants are nonnegative. Thus, the size of $\mathcal{G}_{m,k}$ is the same as the size of

$$\mathcal{I}_{m,k} := \{(\mathrm{sgn}(p(\mathcal{A},B;x^{(1)})),\ldots,\mathrm{sgn}(p(\mathcal{A},B;x^{(n)}))) : \mathcal{A} \in (\mathbb{S}_k^m)^d, B \in \mathbb{S}_k^m\}),$$

where for each i, $p(\mathcal{A},B;x^{(i)}) := \min\{p_1(\mathcal{A},B;x^{(i)}),\ldots,p_m(\mathcal{A},B;x^{(i)})\}$ is a piecewise polynomial in $\mathcal{A}$ and B. To bound the size of $\mathcal{I}_{m,k}$, we use the idea from [1]. We can partition $(\mathbb{S}_k^m)^{d+1}$ into at most mn regions over which the vector is coordinatewise a fixed polynomial. Then we apply Proposition 2.3.

We have n polynomials of degree at most m in up to $D = (d+1)km$ variables, i.e., the number of degrees of freedom of $d+1$ $m \times m$ k-block matrices. Thus, the number of distinct sign vectors in $\mathcal{I}_m$ satisfies $|\mathcal{I}_m| \leq 2mn(\frac{2en}{(d+1)k})^{(d+1)km}$. This implies that $2^\rho \leq 2mn(\frac{2en}{(d+1)k})^{(d+1)km}$, and hence

$$\rho \leq \frac{(d+1)km}{\log 2} \log\left(\frac{2en}{(d+1)k}\right) + \frac{\log(2mn)}{\log 2} \leq c_1 km(d+1)\log\left(\frac{c_2 n}{k}\right).$$

The second claim follows similarly, where instead $D = dkm$. ∎

We can now obtain an upper bound on the risk of the estimators $\hat{f}_{m,k}$ and $\hat{K}_{m,k}$. Recall that we assume f_* and functions in $\mathcal{F}_{m,k}$ are uniformly bounded by some $\Gamma \in (0,\infty)$, and for support function estimation we assume K_* and elements of $\mathcal{L}(\mathcal{S}_{m,k})$ are contained in $B_d(0,\Gamma)$.

THEOREM 2.4.

(i) *For any convex function $f_* : \mathbb{R}^d \to \mathbb{R}$, there exist absolute constants $c, b > 0$ such that*

$$\mathbb{E}\left[\|\hat{f}_{m,k} - f_*\|_\mu^2\right] \leq c\left(\inf_{f\in\mathcal{F}_{m,k}} \|f - f_*\|_\mu^2 + \max\{\sigma^2,\Gamma^2\}km(d+1)\frac{\log(bn/k)}{n}\right).$$

(ii) *For any convex body K_* in $\mathbb{R}^d$, there exist absolute constants $c, b > 0$ such that*

$$\mathbb{E}\left[\ell_\nu^2(\hat{K}_{m,k}, K_*)\right] \leq c\left(\inf_{S\in\mathcal{L}(\mathcal{S}_{m,k})} \ell_\nu^2(S,K_*) + \max\{\sigma^2,\Gamma^2\}kmd\frac{\log(bn/k)}{n}\right)$$

Proof. This result follows from Theorem 2.1 and Lemma 2.2. ∎

Remark 2.5. Theorem 4.2 in [11] also provides high probability tail bounds for the risk that could also be applied here to obtain high probability statements for the risk of spectrahedral estimators.

2.2. Minimax rates. The minimax risk for estimating a function in the class $\mathcal{F}$ from $\{x^{(i)}, y^{(i)}\}_{i=1}^n$ in the random design setting is defined by

$$R_\mu(n,\mathcal{F}) := \min_{\hat{f}} \max_{f\in\mathcal{F}} \mathbb{E}[\|\hat{f} - f\|_\mu].$$

In Table 1 we summarize known rates as $n \to \infty$ of this minimax risk for certain subclasses of convex functions. First consider the class $\mathcal{F}_{m,k}(\Omega)$ of functions in $\mathcal{F}_{m,k}$ with compact and convex domain $\Omega \subset \mathbb{R}^d$. In this case, the rate of convergence is $O(\frac{\log n}{n})$ when the domain Ω satisfies a certain smoothness assumption (see [11, Theorem 2.6]), where we appeal to the fact that $\mathcal{F}_{m,1} \subseteq \mathcal{F}_{m,k}$. Otherwise, the best lower bound on the risk is $O\left(\frac{1}{n}\right)$ using standard arguments for parametric estimation.

Additionally we consider two nonparametric subclasses of convex functions. First is Lipschitz convex regression, where we assume the true function f_* belongs to the class $\mathcal{C}_L(\Omega)$ of L-Lipschitz convex functions with convex and compact full-dimensional

TABLE 1
Minimax rates for subclasses of convex functions.

$\mathcal{F}$	$\mathcal{F}_{m,k}(\Omega)$, for Ω smooth [11]	$\mathcal{C}_L(\Omega)$ [28]	$\mathcal{K}(\Gamma)$ [10]
$R_\mu(n,\mathcal{F})$	$\frac{\log n}{n}$	$n^{-\frac{4}{d+4}}$	$n^{-\frac{4}{d+3}}$

support $\Omega \subset \mathbb{R}^d$. Second is support function estimation, where we assume the true function is the support function of a set K belonging to the collection $\mathcal{K}(\Gamma)$ of convex and compact subsets of $\mathbb{R}^d$ contained in the ball $B_d(0,\Gamma)$ for some finite $\Gamma > 0$. In both settings, the usual least-squares estimator (LSE) over the whole class is minimax suboptimal [16, 17] for all d large enough, necessitating a regularized LSE to obtain the minimax rate.

2.3. Approximation rates. For Lipschitz convex regression, Lemma 4.1 in [28] implies the following: for $f_* \in \mathcal{C}_L(\Omega)$,

$$\inf_{f\in\mathcal{F}_{m,1}} \|f - f_*\|_\mu \le \inf_{f\in\mathcal{F}_{m,1}} \|f - f_*\|_\infty \le c_{d,\Omega,L} m^{-2/d}. \tag{2.5}$$

For support function estimation, let $d_H(S,K) := \|h_S - h_K\|_\infty$ denote the Hausdorff distance between any S and K in $\mathcal{K}$. A classical result of Bronstein (see section 4.1 in [3]) implies

$$\inf_{S\in\mathcal{L}(\mathcal{S}_{m,1})} \ell_\nu(S,K) \le \inf_{\mathcal{L}(\mathcal{S}_{m,1})} d_H(S,K) \le c_{d,\Gamma} m^{-2/(d-1)}. \tag{2.6}$$

This result is also the core of the proof of (2.5).

We first show that inserting (2.5) and (2.6) into Theorem 2.4 and optimizing over m gives general upper bounds on the risk for our (m,k)-spectrahedral estimators. These rates match the minimax rate up to logarithmic factors for fixed $k > 0$, and even when k is allowed to depend logarithmically on m.

COROLLARY 2.6. *Suppose $k_m = h(m)$ for a nondecreasing and differentiable function $h : \mathbb{R} \to (0,m]$.*

(a) *(Lipschitz convex regression) Suppose $f_* \in \mathcal{C}_L(\Omega)$ and define the function*

$$g(m) := h'(m) m^{\frac{2d+4}{d}} + h(m) m^{\frac{d+4}{d}}.$$

Then, for $\alpha_n = g^{-1}\left(\frac{2n}{d(d+1)\max\{\sigma^2,\Gamma^2\}\log(bn)}\right)$,

$$\inf_{m\ge 1} \mathbb{E}\left[\|\hat{f}_{m,k_m} - f_*\|_\mu^2\right] \le c_{d,\Omega,\Gamma}\left(\alpha_n^{-\frac{4}{d}} + \max\{\sigma^2,\Gamma^2\}(d+1)\alpha_n h(\alpha_n)\frac{\log(bn)}{n}\right), \tag{2.7}$$

(b) *(Support function estimation) Suppose $K_* \in \mathcal{K}(\Gamma)$ and define the function*

$$g(m) := h'(m) m^{\frac{2(d+1)}{d-1}} + h(m) m^{\frac{d+3}{d-1}}.$$

Then, for $\alpha_n = g^{-1}\left(\frac{2n}{(d-1)d\max\{\sigma^2,\Gamma^2\}\log(bn)}\right)$,

$$\inf_{m\ge 1} \mathbb{E}\left[\ell_\nu^2(\hat{K}_{m,k_m}, K_*)\right] \le c_{d,\Gamma}\left(\alpha_n^{-\frac{4}{d-1}} + \max\{\sigma^2,\Gamma^2\}(d+1)\alpha_n h(\alpha_n)\frac{\log(bn)}{n}\right). \tag{2.8}$$

We now provide two specific examples for particular functions h:

(i) If $h(m) = km^r$ for fixed $k > 0$ and $r \in [0,1]$, then

$$\inf_{m\ge 1} \mathbb{E}\left[\|\hat{f}_{m,k_m} - f_*\|_\mu^2\right] \le O\left(n^{-\frac{4}{(r+1)d+4}} \log(bn)^{\frac{4}{(r+1)d+4}}\right),$$

and

$$\inf_{m\geq 1} \mathbb{E}\left[\ell_\nu^2(\hat{K}_{m,k_m}, K_*)\right] \leq O\left(n^{-\frac{4}{(r+1)(d-1)+4}} \log(bn)^{\frac{4}{(r+1)(d-1)+4}}\right).$$

(ii) If $h(m) = \log m$, then $\alpha_n = O\left(n^{\frac{d}{d+4}} \log(n)^{-\frac{2d}{d+4}}\right)$, and

$$\inf_{m\geq 1} \mathbb{E}\left[\|\hat{f}_{m,k_m} - f_*\|_\mu^2\right] \leq O\left(n^{-\frac{4}{d+4}} \log(n)^{\frac{8}{d+4}}\right),$$

and

$$\inf_{m\geq 1} \mathbb{E}\left[\ell_\nu^2(\hat{K}_{m,k_m}, K_*)\right] \leq O\left(n^{-\frac{4}{d+3}} \log(n)^{\frac{8}{d+3}}\right).$$

Indeed, the inverse of $p(x) := x^a \log(x)$ is $p^{-1}(x) = (\frac{ax}{W(ax)})^{1/a}$, where W is the Lambert W function. The bound then follows from the fact that W satisfies $\log W(x) = \log x - W(x)$ and as $x \to \infty$, $W(x) \sim \log(x)$.

Remark 2.7. For the case $k = 1$, Corollary 2.6 recovers the results in [10] and [11], showing that these estimators obtain the minimax rate (up to logarithmic factors) for the relevant class of functions.

Proof. We prove (2.7), and the second statement follows by a similar argument. By Theorem 2.6 and (2.5),

$$\mathbb{E}\left[\|\hat{f}_{m,k_m} - f_*\|_\mu^2\right] \leq c_{d,\Omega,L}\left(m^{-4/d} + \frac{\max\{\sigma^2, \Gamma^2\}(d+1)h(m)m}{n} \log(bn)\right).$$

The $m_\star$ that minimizes the expression in the parentheses above satisfies

$$0 = -\frac{4}{d}(m_\star)^{-\frac{4}{d}-1} + \frac{\max\{\sigma^2, \Gamma^2\}(d+1)\log(bn)}{n}\left(h'(m_\star)m_\star + h(m_\star)\right),$$

or, equivalently,

$$\frac{4n}{d(d+1)\max\{\sigma^2, \Gamma^2\}\log(bn)} = h'(m_\star)m_\star^{\frac{2d+4}{d}} + h(m_\star)m_\star^{\frac{d+4}{d}} = g(m_\star).$$

Then, $m_\star = g^{-1}(\frac{4n}{d(d+1)\max\{\sigma^2,\Gamma^2\}\log(bn)})$, and plugging this back into the upper bound gives the result. ∎

As stated previously, an important observation from Corollary 2.6 is that when $k_m = k$ is a fixed constant that does not depend on m, the risk bounds for an optimal choice $m_\star$ match (up to logarithmic factors) the minimax lower bounds of the classes $\mathcal{C}_L(\Omega)$ and $\mathcal{K}(\Gamma)$. This indicates that the approximation rate for the classes $\mathcal{F}_{m,k}$ and $\mathcal{L}(\mathcal{S}_{m,k})$ for fixed k cannot be improved from the rate inherited from the subclasses $\mathcal{F}_{m,1}$ and $\mathcal{L}(\mathcal{S}_{m,1})$, respectively. Indeed, this statistical risk analysis provides the following main result of this section: approximation rate lower bounds for the parametric classes $\mathcal{F}_{m,k}$ and $\mathcal{L}(\mathcal{S}_{m,k})$.

THEOREM 2.8. *Suppose there exists an absolute constant $c > 0$ and $t \in [0,1]$ such that $k_m \leq cm^t$ for all m large enough. Let $f_* \in \mathcal{C}_L(\Omega)$. For all $\varepsilon > 0$, for all m large enough,*

$$\inf_{f\in\mathcal{F}_{m,k_m}} \|f - f_*\|_\infty \geq c_{d,L,\Omega} m^{-2(1+t)/d-\varepsilon}.$$

Also, let $K_* \in \mathcal{K}(\Gamma)$. For all $\varepsilon > 0$, for all m large enough,

$$\inf_{S \in \mathcal{L}(\mathcal{S}_{m,k_m})} d_H(S, K_*) \geq c_{d,\Gamma} m^{-2(1+t)/(d-1)-\varepsilon}.$$

Remark 2.9. For constant k (i.e., $t = 0$), Theorem 2.8 implies

$$\inf_{f \in \mathcal{F}_{m,k}} \|f - f_*\|_\infty = \tilde{O}(m^{-2/d}) \text{ and } \inf_{S \in \mathcal{L}(\mathcal{S}_{m,k})} d_H(S, K_*) = \tilde{O}(m^{-2/(d-1)}),$$

where the $\tilde{O}$ notation ignores polylogarithmic factors.

Proof. We argue by contradiction. Suppose there is some $r > \frac{4}{d}(1+t)$ such that for all $m > 0$,

$$\inf_{f \in \mathcal{F}_{m,k}} \|f - f_*\|_\mu^2 \leq c_1 m^{-r},$$

for some constant c_1 (that may depend on L and Ω). Then by Theorem 2.4 and the minimax lower bound for $\mathcal{C}_L(\Omega)$, there exist constants c_2, b such that for all n large enough,

$$n^{-4/(d+4)} \leq c_2 \inf_{m>0} \left(m^{-r} + \max\{\sigma^2, \Gamma^2\} m^{t+1} (d+1) \frac{\log(bn)}{n} \right).$$

The infimum on the right-hand side is achieved at $m_\star = \left(\frac{rn}{\max\{\sigma^2,\Gamma^2\}(d+1)\log(bn)}\right)^{\frac{1}{t+r+1}}$, and thus

$$n^{-4/(d+4)} \leq c_2 n^{-\frac{r}{t+r+1}} \log(bn)^{\frac{r}{t+r+1}} (\max\{\sigma^2, \Gamma^2\}(d+1))^{\frac{r}{t+r+1}} \left[r^{\frac{-r}{t+r+1}} + r^{\frac{t+1}{t+r+1}} \right].$$

For this inequality to hold for all n, it must hold that $r \leq \frac{4}{d}(1+t)$, a contradiction. The second statement is proved similarly. ∎

3. Computational guarantees.

3.1. Alternating minimization algorithm.

We now describe an alternating minimization algorithm to solve the nonconvex optimization problem (1.3). Let $\xi^{(i)} = (x^{(i)}, 1) \in \mathbb{R}^{d+1}$ for each $i = 1, \ldots, n$, and let $\mathcal{A}_* \in (\mathbb{S}_k^m)^{d+1}$ be the true underlying parameters. That is, we assume our observations for each $i = 1, \ldots, n$ satisfy

$$y_i = \lambda_{\max}(\mathcal{A}_*[\xi^{(i)}]) + \varepsilon_i.$$

We assume the ε_i's are i.i.d. mean zero Gaussian noise with variance σ^2.

One iteration of the algorithm starts with a fixed parameter $\mathcal{A} \in (\mathbb{S}_k^m)^{d+1}$ and proceeds as follows. We first compute the maximizing eigenvector $u^{(i)} \in S^{m-1}$ for each $i = 1, \ldots, n$, such that for $U^{(i)} = u^{(i)}(u^{(i)})^T$, $\langle U^{(i)}, \mathcal{A}[\xi^{(i)}]\rangle = \lambda_{\max}(\mathcal{A}[\xi^{(i)}])$. With the $U^{(i)}$'s fixed, the second step is to update $\mathcal{A}$ by solving the linear least squares problem

$$\mathcal{A}^+ \in \operatorname{argmin}_{\mathcal{A} \in (\mathbb{S}_k^m)^{d+1}} \frac{1}{n} \sum_{i=1}^n \left(y^{(i)} - \langle U^{(i)}, \mathcal{A}[\xi^{(i)}]\rangle \right)^2, \tag{3.1}$$

where $\langle U^{(i)}, \mathcal{A}[\xi^{(i)}]\rangle = \langle \mathcal{A}, \xi^{(i)} \otimes U^{(i)}\rangle = \sum_{j=1}^d \langle A_j, \xi_j^{(i)} U^{(i)}\rangle$. Note that in the algorithm description below, Step 2 implicitly depends on k because if $\mathcal{A} \in (\mathbb{S}_k^m)^{d+1}$, then $\mathcal{A}^+$ will also be in $(\mathbb{S}_k^m)^{d+1}$.

Algorithm 3.1 Alternating Minimization for Spectrahedral Regression

Input: Collection of inputs and outputs $\{(x^{(i)}, y^{(i)})\}_{i=1}^n$; initialization $\mathcal{A} \in (\mathbb{S}_k^m)^{d+1}$

Algorithm: Repeat until convergence

Step 1: Update optimal eigenvector $u^{(i)} \leftarrow \lambda_{\max}(\mathcal{A}[\xi^{(i)}])$

Step 2: Update $\mathcal{A}$ by solving (3.1). $\mathcal{A}^+ \leftarrow (\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}})^{-1} \Xi_{\mathcal{A}}^T y$, where $\Xi_{\mathcal{A}}^T = (\xi^{(1)} \otimes U^{(1)} | \cdots | \xi^{(n)} \otimes U^{(n)}) \in \mathbb{R}^{(d+1)m^2 \times n}$.

Output: Final iterate $\mathcal{A}$

3.2. Convergence guarantee. The following result shows that under certain conditions, this alternating minimization procedure converges geometrically to a small ball around the true parameters given a good initialization. To state the initialization condition in the result, we define for $\mathcal{A} \in (\mathbb{S}_k^m)^d$ the similarity transformation $\mathcal{O}(\mathcal{A}) = (OA_1O^T, \ldots, OA_dO^T)$ for an orthogonal $m \times m$ matrix O with blocks of size k. Note that the eigenvalues of $\mathcal{A}[x]$ for $x \in \mathbb{R}^d$ are invariant under any such $\mathcal{O}$.

The proof of the following result appears after the statement, and it depends on multiple lemmas that we state and prove in the appendix.

THEOREM 3.1. *Assume $X \sim \mu$ is an η-sub-Gaussian random vector in $\mathbb{R}^d$ such that $\mathbb{E}[X_i^2] = 1$ for each $i = 1, \ldots, d$. Also suppose that the true parameter $\mathcal{A}_* \in (\mathbb{S}_k^m)^{d+1}$ satisfies the following spectral condition:*

$$\inf_{u \in S^d} \lambda_1(\mathcal{A}_*[u]) - \lambda_2(\mathcal{A}_*[u]) := \kappa > 0, \tag{3.2}$$

where $\lambda_1 := \lambda_{\max}$ and λ_2 is the second largest eigenvalue. Let $\tilde{\eta} := \max\{\eta, 1\}$ and fix $\tau \in (0,1)$. Then, there exist constants c_i, $i = 1, \ldots, 4$, such that if the initial parameter choice $\mathcal{A}_0$ satisfies

$$\|\mathcal{A}_0 - \mathcal{O}(\mathcal{A}_*)\|_F^2 \leq \frac{3\kappa^2}{128(d+1)m} \left(\frac{1-\tau}{1+\tau}\right),$$

for some similarity transformation $\mathcal{O}$ and

$$n \geq c_1 m^3 \max\left\{ \left(\frac{1+\tau}{1-\tau}\right) \frac{(d+1)^2 m \sigma^2 \log(n)^2}{\kappa^2(1-\tau)}, \tau^{-2}\tilde{\eta}^6(d+1)m, \tilde{\eta}^2 \max\{1, \sigma^2\} \right\},$$

then the error at all iterations t simultaneously satisfies

$$\|\mathcal{A}_t - \mathcal{O}(\mathcal{A}_*)\|_F^2 \leq \left(\frac{3}{4}\right)^t \|\mathcal{A}_0 - \mathcal{O}(\mathcal{A}_*)\|_F^2 + \frac{c_2 m^3 (d+1)\sigma^2 \log(n)^2}{n(1-\tau)},$$

with probability greater than $1 - 6\exp\{-c_3\tau^2 n/(\tilde{\eta}^6(d+1)m^4)\} - n^{-c_4 m}$.

Before proceeding, we provide some examples of parameters $\mathcal{A} = (A_1, \ldots, A_{d+1}) \in (\mathbb{S}^m)^{d+1}$ where assumption (3.2) is satisfied.

First, consider the case where $d = 2$, $m = 2$, and $A_3 = 0$. Note that for $u \in \mathbb{S}^{d-1}$, there are vectors $a_{ij} \in \mathbb{R}^2$ for $i, j = 1, 2$ such that

$$\lambda_1(\mathcal{A}[u]) = \lambda_1 \begin{bmatrix} \langle a_{11}, u \rangle & \langle a_{12}, u \rangle \\ \langle a_{12}, u \rangle & \langle a_{22}, u \rangle \end{bmatrix} = \langle a_{11} + a_{22}, u \rangle + \sqrt{\langle a_{11} - a_{22}, u \rangle^2 + 4\langle a_{12}, u \rangle^2}$$

and

$$\lambda_2(\mathcal{A}[u]) = \lambda_2 \begin{bmatrix} \langle a_{11}, u \rangle & \langle a_{12}, u \rangle \\ \langle a_{12}, u \rangle & \langle a_{22}, u \rangle \end{bmatrix} = \langle a_{11} + a_{22}, u \rangle - \sqrt{\langle a_{11} - a_{22}, u \rangle^2 + 4\langle a_{12}, u \rangle^2}.$$

Then, the eigengap satisfies

$$\lambda_1 \begin{bmatrix} \langle a_{11}, u\rangle & \langle a_{12}, u\rangle \\ \langle a_{12}, u\rangle & \langle a_{22}, u\rangle \end{bmatrix} - \lambda_2 \begin{bmatrix} \langle a_{11}, u\rangle & \langle a_{12}, u\rangle \\ \langle a_{12}, u\rangle & \langle a_{22}, u\rangle \end{bmatrix}$$
$$= 2\sqrt{\langle a_{11} - a_{22}, u\rangle^2 + 4\langle a_{12}, u\rangle^2} = \|\tilde{A}u\|_2 \geq \sigma_{\min}(\tilde{A}),$$

where $\tilde{A} = \left[\begin{smallmatrix} 2(a_{11}-a_{22}) \\ 4a_{12} \end{smallmatrix}\right]$. Thus, if $\tilde{A}$ has a positive minimum singular value, (3.2) will hold.

Another example of a parameter $\mathcal{A}$ that satisfies condition (3.2) is when $\lambda_{\max}(\mathcal{A}[x]) = \|x\|_2$. This is the parameter $\mathcal{A} = (A_1, \ldots, A_{d+1}) \in (\mathbb{S}^{d+1})^{d+1}$ such that

$$(A_i)_{jk} = \begin{cases} 1, & j = 1, k = i+1 \text{ or } k = 1, j = i+1, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, d$ and $A_{d+1} = 0$. Indeed, we see that

$$f(x) = \lambda_{\max}(\mathcal{A}[x]) = \lambda_{\max}\left(\begin{bmatrix} 0 & x_1 & \cdots & x_d \\ x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_d & 0 & \cdots & 0 \end{bmatrix}\right) = \|x\|_2. \tag{3.3}$$

In fact, for any spectrahedral function $f(x) = \lambda_{\max}(\mathcal{A}[x])$ that is differentiable for all $x \in \mathbb{S}^{d-1}$, $\mathcal{A}$ must necessarily satisfy (3.2).

However, there are examples that do not satisfy assumption (3.2). In particular, it will never be satisfied in the setting of support function estimation when $k = 1$, because the eigengap will achieve the minimum value of zero for u in the directions of the vertices of the associated polytope. In the next result, we provide a second convergence guarantee with a weaker condition on $\mathcal{A}_*$ at the expense of stronger conditions on the initialization and the covariate distribution μ as well as a weaker bound in probability. Following the statement we will describe examples when the condition is satisfied in the $k = 1$ setting.

To state the conditions in the following result, we denote by $\mathcal{A}_*^{(j)}$ for $j = 1, \ldots m/k$ the $(d+1)$-tuples of $k \times k$ symmetric matrices that make up the blocks of the $(d+1)$-tuple $\mathcal{A}_*$.

THEOREM 3.2. *Let $X \sim \mu$ be a random vector in $\mathbb{R}^d$ such that $\|X\|_\infty \leq \eta$ and μ is a continuous distribution. Define $\xi := (X, 1) \in \mathbb{R}^{d+1}$. Assume that there is a constant $c > 0$ such that for all $\mathcal{A} \neq \mathcal{B} \in (\mathbb{S}^k)^{d+1}$,*

$$\mathbb{P}\left(|\lambda_1(\mathcal{A}[\xi]) - \lambda_1(\mathcal{B}[\xi])| \leq \rho \mathbb{E}\left[|\lambda_1(\mathcal{A}[\xi]) - \lambda_1(\mathcal{B}[\xi])|\right]\right) \leq c\rho \quad \text{for all } \rho > 0. \tag{3.4}$$

Also assume there exist $\kappa > 0$ and $\delta \in (0,1)$ such that

$$\inf_{j,\ell \in \{1,\ldots,m/k\}: j \neq \ell} \mathbb{E}\left[|\lambda_1(\mathcal{A}_*^{(j)}[\xi]) - \lambda_1(\mathcal{A}_*^{(\ell)}[\xi])|\right] \geq \frac{m\kappa}{k\delta}, \tag{3.5}$$

and additionally, if $k \geq 2$,

$$\inf_{j=1,\ldots,m/k} \inf_{u \in S^d} \lambda_1(\mathcal{A}_*^{(j)}[u]) - \lambda_2(\mathcal{A}_*^{(j)}[u]) := \kappa > 0. \tag{3.6}$$

Let $\tilde{\eta} := \max\{\eta, 1\}$ and fix $\tau \in (0,1)$. Then, there exist constants c_i, $i = 1, \ldots, 4$, such that if the initial parameter choice $\mathcal{A}_0$ satisfies

$$\|\mathcal{A}_0 - \mathcal{O}(\mathcal{A}_*)\|_F \leq \frac{3k^{3/2}\kappa}{256 \max\{1, c\}\tilde{\eta}^3 (d+1)^{3/2} m^{5/2}} \left(\frac{1-\tau}{1+\tau}\right), \tag{3.7}$$

for some similarity transformation $\mathcal{O}$ *and*

$$n \geq c_1 \max\left\{\frac{\max\{1,c^2\}\tilde{\eta}^6(d+1)^4 m^8(1-\tau)\sigma^2\log(n)^2}{\kappa^2 k^3(1+\tau)^2}, \frac{mk\tilde{\eta}^3(d+1)^{3/2}}{\tau^2}\right\},$$

then the error at all iterations $t \geq 1$ *simultaneously satisfies*

$$\|\mathcal{A}_t - \mathcal{O}(\mathcal{A}_*)\|_F^2 \leq \left(\frac{3}{4}\right)^t \|\mathcal{A}_0 - \mathcal{O}(\mathcal{A}_*)\|_F^2 + \frac{c_2 m^3(d+1)\sigma^2\log(n)^2}{n(1-\tau)},$$

with probability greater than $1 - 3c\delta - n^{-\frac{c_3 m}{\tilde{\eta}^2}\min\left\{\frac{(d+1)m}{\sigma^2},1\right\}} - 6de^{-\frac{c_4 n}{\tilde{\eta}^2(d+1)m}}$.

Remark 3.3. Condition (3.4) is an example of a small-ball property for random vectors that appears in the probability literature; see, for instance, [20, 24]. A small-ball condition also appears in [9], which considers the polyhedral setting.

To see an example when these conditions are satisfied for the case $k=1$, consider the setting of support function estimation. The covariates are unit vectors $u^{(i)}$ on S^{d-1} and the parameter space is $(\mathbb{S}_k^m)^d$. Let $k=1$, and assume the covariates are i.i.d. samples of a random unit vector U. Then, condition (3.4) is equivalent to the following: for all $a \neq b \in \mathbb{R}^d$,

$$\mathbb{P}\left(|\langle a-b, U\rangle| \leq \rho\mathbb{E}\left[|\langle a-b,U\rangle|\right]\right) \leq c\rho \quad \text{ for all } \rho > 0.$$

When U is uniform on the unit sphere S^{d-1}, this is satisfied. Indeed, letting σ denote the normalized spherical Lesbesgue measure on S^{d-1} and ω_d denote the surface area of S^{d-1}, first observe that

$$\begin{aligned}\mathbb{E}|\langle a-b,U\rangle| = \|a-b\|_2\mathbb{E}|U_1| &= \|a-b\|_2\int_{S^{d-1}}|u_1|\mathrm{d}\sigma(u)\\ &= \frac{\|a-b\|_2\omega_{d-1}}{\omega_d}\int_{-1}^1 |t|(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t = c_1\|a-b\|_2,\end{aligned}$$

for a finite constant $c_1 > 0$ depending on d. Then, for all $\rho > 0$,

$$\begin{aligned}\mathbb{P}\left(|\langle a-b,U\rangle| \leq \rho\mathbb{E}\left[|\langle a-b,U\rangle|\right]\right) &= \mathbb{P}\left(|\langle a-b,U\rangle| \leq \rho c_1\|a-b\|_2\right)\\ &= \int_{S^{d-1}} 1_{\{|u_1|\leq\rho c_1\}}\mathrm{d}\sigma(u)\\ &= \frac{2\omega_{d-1}}{\omega_d}\int_0^{\min\{\rho c_1,1\}}(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t \leq c_2\rho,\end{aligned}$$

where $c_2 > 0$ depends only on d. Second, condition (3.5) is satisfied when

$$\inf_{\ell\neq j} c_1\|a_*^{(\ell)} - a_*^{(j)}\|_2 > m\kappa/\delta,$$

where $a_*^{(j)} = ((A_1)_{jj}, \ldots (A_d)_{jj})$ for each $j = 1, \ldots m$. The final condition (3.6) is not relevant in the case where $k=1$, but blocks of the form given by examples following Theorem 3.1 will satisfy this condition.

3.3. Proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. First, given assumption (3.2), we show that for n large enough, for all parameters $\mathcal{A}$ satisfying for some similarity transform $\mathcal{O}$,

$$\|\mathcal{A} - \mathcal{O}(\mathcal{A}_*)\|_F^2 \leq \frac{3\kappa^2}{128(d+1)m}\left(\frac{1-\tau}{1+\tau}\right), \tag{3.8}$$

the parameter $\mathcal{A}^+$ obtained after applying one iteration of the algorithm satisfies

$$\|\mathcal{A}^+ - \mathcal{O}(\mathcal{A}_*)\|_F^2 \le \frac{3}{4}\|\mathcal{A} - \mathcal{O}(\mathcal{A}^*)\|_F^2 + O\left(\frac{\log(n)^2}{n}\right) \tag{3.9}$$

with high probability.

Let $U^{(i)} = u^{(i)}(u^{(i)})^T$ be such that $\lambda_{\max}(\mathcal{A}[\xi^{(i)}]) = \langle U^{(i)}, \mathcal{A}[\xi^{(i)}]\rangle$. The update $\mathcal{A}^+$ then equals

$$\mathcal{A}^+ = (\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}})^{-1}\Xi_{\mathcal{A}}^T y,$$

where

$$\Xi_{\mathcal{A}}^T = (\xi^{(1)} \otimes U^{(1)}| \cdots |\xi^{(n)} \otimes U^{(n)}) \in \mathbb{R}^{(d+1)m\times m\times n}. \tag{3.10}$$

Note that $(\Xi_{\mathcal{A}}\mathcal{A})_i = \langle U^{(i)}, \mathcal{A}[\xi^{(i)}]\rangle$. Throughout the rest of the proof, we sometimes abuse notation and consider the Kronecker product $\xi \otimes U$ for $\xi \in \mathbb{R}^{d+1}$ and $U \in \mathbb{R}^{m\times m}$ to be the vector $\mathrm{Vec}(\xi \otimes U) \in \mathbb{R}^{(d+1)m^2}$.

By the invariance $\lambda_{\max}(\mathcal{A}[x]) = \lambda_{\max}(\mathcal{O}(\mathcal{A})[x])$ for all $\mathcal{O}$, without loss of generality we can assume in the following that $\mathcal{A}_* = \mathcal{O}(\mathcal{A}_*)$ for the transformation $\mathcal{O}$ satisfying assumption (3.8). Let $y_* \in \mathbb{R}^n$ and $u_*^{(i)} \in \mathbb{S}^{d-1}$ be such that for $U_*^{(i)} := u_*^{(i)}(u_*^{(i)})^T$,

$$y_i^* = \langle U_*^{(i)}, \mathcal{A}_*[\xi^{(i)}]\rangle = \lambda_{\max}(\mathcal{A}_*[\xi^{(i)}]).$$

Also denote by $P_{\Xi_{\mathcal{A}}} = \Xi_{\mathcal{A}}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})^{-1}\Xi_{\mathcal{A}}^T$ the orthogonal projection onto the span of the columns of $\Xi_{\mathcal{A}}$. Then, we have the following deterministic upper bound:

$$\begin{aligned}\|\Xi_{\mathcal{A}}(\mathcal{A}^+ - \mathcal{A}_*)\|^2 &= \|P_{\Xi_{\mathcal{A}}}y - \Xi_{\mathcal{A}}\mathcal{A}_*\|^2 = \|P_{\Xi_{\mathcal{A}}}y^* + P_{\Xi_{\mathcal{A}}}\varepsilon - \Xi_{\mathcal{A}}\mathcal{A}_*\|^2 \\ &\le 2\|P_{\Xi_{\mathcal{A}}}(y^* - \Xi_{\mathcal{A}}\mathcal{A}_*)\|^2 + 2\|P_{\Xi_{\mathcal{A}}}\varepsilon\|^2 \\ &\le 2\sum_{i=1}^n \left(\langle U_*^{(i)}, \mathcal{A}_*[\xi^{(i)}]\rangle - \langle U^{(i)}, \mathcal{A}_*[\xi^{(i)}]\rangle\right)^2 + 2\|P_{\Xi_{\mathcal{A}}}\varepsilon\|^2.\end{aligned}$$

Now, since $\langle U^{(i)} - U_*^{(i)}, \mathcal{A}[\xi^{(i)}]\rangle \ge 0$,

$$\begin{aligned}&\left(\langle U_*^{(i)}, \mathcal{A}_*[\xi^{(i)}]\rangle - \langle U^{(i)}, \mathcal{A}_*[\xi^{(i)}]\rangle\right)^2 \\ &\quad\le \left(\langle U_*^{(i)} - U^{(i)}, \mathcal{A}_*[\xi^{(i)}]\rangle + \langle U^{(i)} - U_*^{(i)}, \mathcal{A}[\xi^{(i)}]\rangle\right)^2 \\ &\quad= \left\langle \mathcal{A} - \mathcal{A}_*, \xi^{(i)} \otimes (U^{(i)} - U_*^{(i)})\right\rangle^2.\end{aligned}$$

We also have the lower bound $\|\Xi_{\mathcal{A}}(\mathcal{A}^+ - \mathcal{A}_*)\|^2 \ge \lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})\|\mathcal{A}^+ - \mathcal{A}_*\|^2$. Thus,

$$\begin{aligned}\|\mathcal{A}^+ - \mathcal{A}_*\|^2 &\le \frac{2}{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}\left[\|\Xi_{\mathcal{A}-\mathcal{A}_*}(\mathcal{A} - \mathcal{A}_*)\|_2^2 + \|P_{\Xi_{\mathcal{A}}}\varepsilon\|^2\right] \\ &\le \frac{2}{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}\left[\lambda_{\max}(\Xi_{\mathcal{A}-\mathcal{A}_*}^T\Xi_{\mathcal{A}-\mathcal{A}_*})\|\mathcal{A} - \mathcal{A}_*\|^2 + \|P_{\Xi_{\mathcal{A}}}\varepsilon\|^2\right]. \quad (3.11)\end{aligned}$$

where $\Xi_{\mathcal{A}-\mathcal{A}_*} = (\xi^{(1)} \otimes (U^{(1)} - U_*^{(1)})| \cdots |\xi^{(n)} \otimes (U^{(n)} - U_*^{(n)}))$.

Next, note that $\xi = (X, 1)$ is $\tilde{\eta}$-sub-Gaussian, where $\tilde{\eta} = \max\{\eta, 1\}$. Lemmas A.2 and A.3 then imply the following. For $\tau \in (0,1)$, there exist absolute constants

c_1, c_2 such that if $n \geq c_1\tau^{-2}\tilde{\eta}^6(d+1)m^4$, then with probability greater than $1 - 2e^{-c_2\tau^2 n/(\tilde{\eta}^6(d+1)m^4)}$,

$$\lambda_{\max}(\Xi_{\mathcal{A}-\mathcal{A}_*}^T \Xi_{\mathcal{A}-\mathcal{A}_*}) \leq n\lambda_{\max}(\mathbb{E}[(\xi \otimes (U-U_*))(\xi \otimes (U-U_*))^T])\,(1+\tau) \tag{3.12}$$

for all $\mathcal{A}$ satisfying assumption (3.8). Since $\lambda_{\max}$ is a convex function, Jensen's inequality implies

$$\lambda_{\max}(\mathbb{E}[(\xi \otimes (U-U_*))(\xi \otimes (U-U_*))^T]) \leq \mathbb{E}[\|\xi \otimes (U-U_*)\|^2].$$

Then, by the definition of the Kronecker product, Lemma A.2, and the assumption on X,

$$\mathbb{E}\left[\|\xi \otimes (U-U_*)\|^2\right] = \mathbb{E}\left[\|\xi\|_2^2\|U-U_*\|_F^2\right] \leq 32\kappa^{-2}(d+1)\|\mathcal{A}-\mathcal{A}_*\|_F^2, \tag{3.13}$$

Putting the bounds together and using assumption (3.8), we have

$$\lambda_{\max}(\mathbb{E}[(\xi \otimes (U-U_*))(\xi \otimes (U-U_*))^T]) \leq 32\kappa^{-2}(d+1)\|\mathcal{A}-\mathcal{A}_*\|_F^2 \leq \frac{3}{8m}\left(\frac{1-\tau}{1+\tau}\right). \tag{3.14}$$

Plugging the bound (3.14) into (3.12) then gives the upper bound

$$\lambda_{\max}(\Xi_{\mathcal{A}-\mathcal{A}_*}^T \Xi_{\mathcal{A}-\mathcal{A}_*}) \leq \frac{3n}{8m}\,(1-\tau)\,. \tag{3.15}$$

Also by Lemmas A.2 and A.3, if $n \geq c_1\tau^{-2}\tilde{\eta}^6(d+1)m^4$, then with probability greater than $1 - 2e^{-c_2\tau^2 n/(\tilde{\eta}^6(d+1)m^4)}$,

$$\lambda_{\min}(\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}}) \geq n\lambda_{\max}(\mathbb{E}[(\xi \otimes U)(\xi \otimes U)^T])\,(1-\tau) \tag{3.16}$$

for all $\mathcal{A}$ satisfying (3.8). By Lemma A.1,

$$\lambda_{\max}(\mathbb{E}[(\xi \otimes U)(\xi \otimes U)^T]) \geq m^{-1}, \tag{3.17}$$

and plugging the bound (3.17) into (3.16) gives

$$\lambda_{\min}(\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}}) \geq nm^{-1}\,(1-\tau)\,. \tag{3.18}$$

Finally, combining (3.15) and (3.18) with (3.11) implies

$$\|\mathcal{A}^+ - \mathcal{A}^*\|_F^2 \leq \frac{3}{4}\|\mathcal{A} - \mathcal{A}^*\|_F^2 + \frac{2m\|P_{\Xi_{\mathcal{A}}}\varepsilon\|_2^2}{n(1-\tau)}.$$

It remains to bound the error term. By Lemma A.4, there exist constants $c_3, \ldots c_6 > 0$ such that for $n \geq c_3 m^3\tilde{\eta}^2 \max\{\tilde{\eta}^4(d+1)m, \max\{1,\sigma^2\}\}$,

$$\|P_{\Xi_{\mathcal{A}}}\varepsilon\|_2^2 \leq c_4 \log(n)^2\sigma^2 m^2(d+1)$$

for all $\mathcal{A}$ satisfying (3.8) with probability greater than $1 - n^{-c_5 m} - 2e^{-c_6 n/(\tilde{\eta}^6(d+1)m^4)}$. This implies that for

$$n \geq c_7 m^3\tilde{\eta}^2 \max\{\tau^{-2}\tilde{\eta}^4(d+1)m, \max\{1,\sigma^2\}\},$$

with probability at least $1 - 6e^{-c_6\tau^2 n/\tilde{\eta}^6(d+1)m^4} - n^{-c_5 m}$,

$$\|\mathcal{A}^+ - \mathcal{A}^*\|_F^2 \leq \frac{3}{4}\|\mathcal{A} - \mathcal{A}^*\|_F^2 + \frac{c_4 m^3(d+1)\sigma^2 \log(n)^2}{n(1-\tau)}.$$

We now show that given the above upper bound, $\mathcal{A}^+$ also satisfies (3.8). Indeed, for

$$n \geq \frac{4}{\kappa^2} \cdot \frac{128m(d+1)}{3}\left(\frac{1+\tau}{1-\tau}\right)\frac{c_4 m^3(d+1)\sigma^2\log(n)^2}{(1-\tau)},$$

we have

$$\frac{c_4 m^3(d+1)\sigma^2\log(n)^2}{n(1-\tau)} \leq \frac{\kappa^2}{4}\cdot\frac{3}{128m(d+1)}\left(\frac{1-\tau}{1+\tau}\right)$$

and thus

$$\|\mathcal{A}^+ - \mathcal{A}_*\|_F^2 \leq \frac{3}{4}\|\mathcal{A} - \mathcal{A}_*\|_F^2 + \frac{c_4 m^3(d+1)\sigma^2\log(n)^2}{n(1-\tau)} \leq \frac{3\kappa^2}{128m(d+1)}\left(\frac{1-\tau}{1+\tau}\right).$$

The final conclusion follows from the fact that after t iterations, when

$$n \geq c_8 m^3 \max\left\{\left(\frac{1+\tau}{1-\tau}\right)\frac{(d+1)^2 m\sigma^2\log(n)^2}{\kappa^2(1-\tau)}, \tau^{-2}\tilde{\eta}^6(d+1)m, \tilde{\eta}^2\max\{1,\sigma^2\}\right\},$$

applying the bound (3.9) t times gives

$$\begin{aligned}\|\mathcal{A}_t - \mathcal{A}_*\|_F^2 &\leq \left(\frac{3}{4}\right)^t\|\mathcal{A}_0 - \mathcal{A}_*\|_F^2 + \frac{c_4 m^3(d+1)\sigma^2\log(n)^2}{n(1-\tau)}\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k \\ &\leq \left(\frac{3}{4}\right)^t\|\mathcal{A}_0 - \mathcal{A}_*\|_F^2 + \frac{c_9 m^3(d+1)\sigma^2\log(n)^2}{n(1-\tau)},\end{aligned}$$

and all t bounds hold simultaneously with probability at least $1-6\exp\{-c_6\tau^2 n/(\tilde{\eta}^6(d+1)m^4)\} - n^{-c_9 m}$. ∎

Proof of Theorem 3.2. The proof follows the same arguments as the proof of the previous theorem, and replacing Lemma A.2 with Lemma A.5 and Lemma A.4 with Lemma A.7.

Also the bound (3.13) is replaced by the following:

$$\begin{aligned}\mathbb{E}\left[\|\xi \otimes (U - U_*)\|^2\right] &= \mathbb{E}\left[\|\xi\|_2^2\|U - U_*\|_F^2\right] \leq 2\tilde{\eta}^2(d+1)\mathbb{E}[\|U - U_*\|_F] \\ &\leq \frac{32\max\{1,c\}\tilde{\eta}^3(d+1)^{3/2}m^{3/2}}{k^{3/2}\kappa}\|\mathcal{A} - \mathcal{A}_*\|_F,\end{aligned}$$

where we use Lemma A.5 and the assumption on X. ∎

4. Numerical experiments. In this section, we empirically compare spectrahedral and polyhedral regression for estimating a convex function from data. More specifically, we compare (m,m)-spectrahedral estimators to $m(m+1)/2$-polyhedral estimators, both of which have $m(m+1)/2$ degrees of freedom per dimension. For each experiment, we apply the alternating minimization algorithm with multiple random initializations, and the solution that minimizes the least squared error is selected. We adapted the code [26] for support function estimation used in [27] for spectrahedral regression.

4.1. Synthetic regression problems. The first experiments use synthetically generated data from a known convex function, one from a spectrahedral function and another from a convex function that is neither polyhedral nor spectrahedral. In both problems below, the root-mean-squared error (RMSE) is obtained by first obtaining estimators from 200 noisy training data points and then evaluating the RMSE of the estimators on 200 test points generated from the true function. We ran the alternating minimization algorithm with 50 random initializations for 200 steps or until convergence, and we chose the best estimator.

First, we consider n i.i.d. data points distributed as (X, Y), where $X \in \mathbb{R}^2$ is uniformly distributed in $[-1,1]^2$, and

$$Y = \sqrt{X_1^2 + X_2^2} + \varepsilon, \tag{4.1}$$

where $\varepsilon \sim \mathcal{N}(0, 0.1^2)$. In Figure 2, we have plotted polyhedral and spectrahedral estimators obtained from $n = 20$, 50, and 200 data points. We have also plotted the least-squares estimator (LSE) in each case. The RMSE for both models is given in Table 2. The function $y = \|x\|_2$ for $x \in \mathbb{R}^2$ is an $m = 3$ spectrahedral function, and the spectrahedral estimator performs better than the polyhedral estimator, as

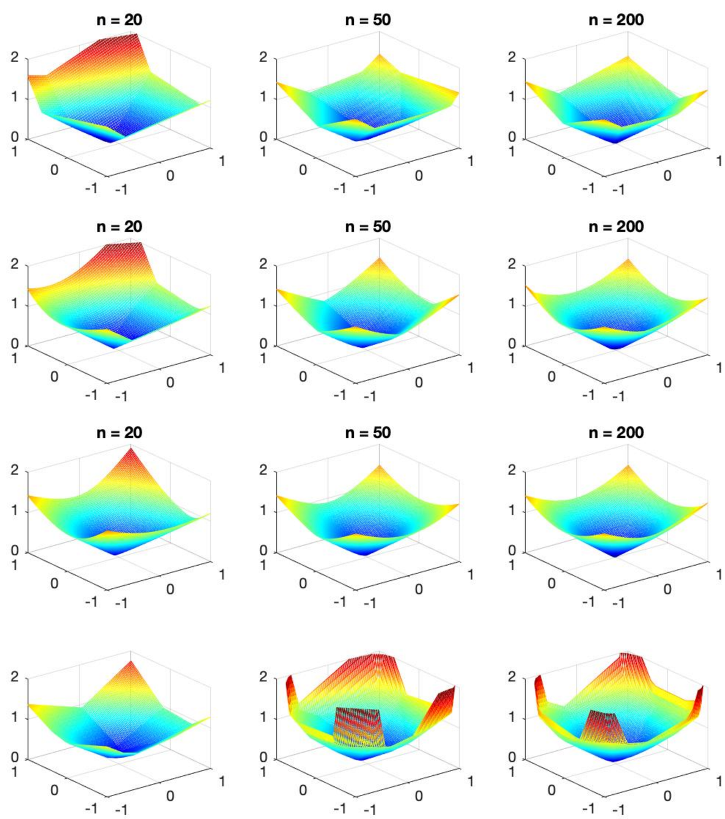

FIG. 2. *From top to bottom: Polyhedral* $((k,m) = (1,6))$*, block spectrahedral* $((k,m) = (2,4))$*, spectrahedral* $(m = 3)$*, and LSE reconstructions of the convex function* $y = \|x\|_2$ *from* $n = 20$*, 50, and* 200 *data points from model* (4.1)*.*

Table 2
RMSE for polyhedral and spectrahedral estimators for data generated from models (4.1) *and* (4.2) *as* m *increases.*

Model	DoF	Spectrahedral $k=m$	Spectrahedral $k=2$	Polyhedral $k=1$
(4.1)	3	0.0143	0.0143	0.1206
	6	0.0261	0.0246	0.0456
	15	0.0290	0.0360	0.0344
(4.2)	3	0.0143	0.0143	0.1206
	6	0.1045	0.3622	0.1358
	15	0.1048	0.3537	0.1266

expected. In addition, the RMSE is lowest for the $m = 3$ spectrahedral estimator and increases for larger m with either $k = m$ or $k = 2$. In the polyhedral case, the RMSE decreases with larger m since the function $y = \|x\|_2$ is not contained in any polyhedral function class. The RMSE for the LSE is 0.8128, which is significantly higher than the polyhedral and spectrahedral estimators. This can be contributed to overfitting, especially near the boundary of the input domain, as illustrated by the plots in Figure 2.

Second, we consider n i.i.d. data points generated as $(X, Y) \in \mathbb{R} \times \mathbb{R}$, where $X \sim \mathcal{N}(0, 1)$ and

$$Y = \exp(bX) + \varepsilon, \tag{4.2}$$

where $b = 1.1394$ and $\varepsilon \sim \mathcal{N}(0, 0.1^2)$. The underlying convex function is neither polyhedral nor spectrahedral, but the spectrahedral estimator better captures the smoothness of the function, as illustrated in Figure 3. The spectrahedral estimator also outperforms the polyhedral estimator with respect to the RMSE; see Table 2. We also plot the LSE obtained from this data set in the last row of Figure 3. The RMSE for the LSE when $n = 200$ is 0.0349, which is smaller that that for the polyhedral and spectrahedral estimators. This shows that overfitting is not as much of a problem here, most likely due to the dimension $d = 1$ of the input.

4.2. Predicting average weekly wages. The first experiment we perform on real data is predicting average weekly wages based on years of education and experience. This data set is also studied in [13]. The data set is from the 1988 Current Population Survey (CPS) and can be obtained as the data set ex1029 in the Sleuth2 package in R. It consists of 25,361 records of weekly wages for full-time, adult, male workers for 1987, along with years of experience and years of education. It is reasonable to expect that wages are concave with respect to the years of experience. Indeed, at first wages increase with more experience, but with a decreasing return each year until a peak of earnings is reached, and then they begin to decline. Wages are also expected to increase as the number of years of education increases, but not in a concave way. However, as in [13], we use the transformation $1.2^{\text{years education}}$ to obtain a concave relationship. We used polyhedral and spectrahedral regression to fit convex functions to this data set, as illustrated in Figure 1. We also estimated the RMSE for different values of $m(m+1)/2$ (the degrees of freedom per dimension) through hold-out validation with 20% of the data points; see Table 3. This generalization error is smaller for the spectrahedral estimator than the polyhedral estimator in each case.

4.3. Convex approximation in engineering applications. In the following two examples, we consider applications of convex regression in engineering

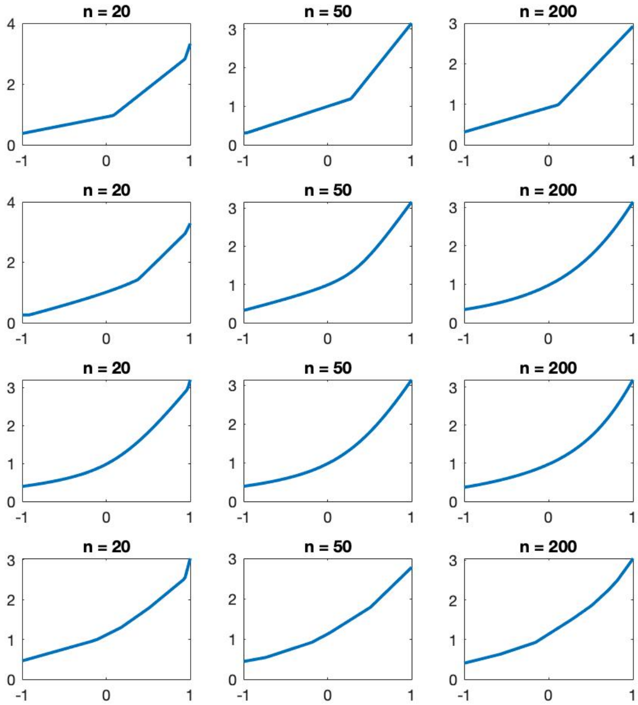

FIG. 3. *From top to bottom: Polyhedral ($(k,m) = (1,6)$), block spectrahedral ($(k,m) = (2,4)$), spectrahedral ($m = 3$), and LSE reconstructions of the convex function $y = \exp(\langle x, b\rangle)$ from $n = 20$, 50, and 200 noisy data points from model* (4.2).

TABLE 3
RMSE for polyhedral and spectrahedral estimators for real data and engineering experiments.

Application	$m(m+1)/2$	Spectrahedral	Polyhedral
Average weekly wages	3	142.1166	145.5803
	6	140.1173	141.4989
	10	140.0352	141.9851
Aircraft profile drag	3	0.086	0.0895
	6	0.0576	0.0709
	10	0.0452	0.0515
Circuit design	3	0.0085	0.02
	6	0.0072	0.012
	10	0.0072	0.0088

applications where the goal is to subsequently use the convex estimator as an objective or constraint in an optimization problem. Polyhedral regression returns a convex function compatible with a linear program, and using spectrahedral regression provides an estimator compatible with semidefinite programming.

4.3.1. Aircraft data. In this experiment, we consider the XFOIL aircraft design problem studied in [14]. The profile drag on an airplane wing is described by a coefficient CD that is a function of the Reynolds number (Re), wing thickness ratio (τ), and lift coefficient (CL). There is not an analytical expression for this relationship, but it can be simulated using XFOIL [5]. For a fixed τ, after a logarithmic

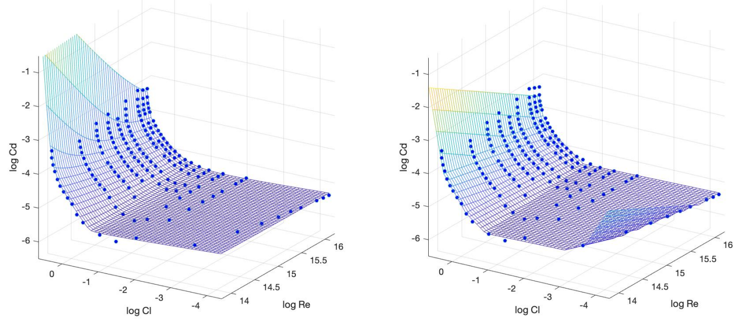

FIG. 4. *Spectrahedral ($m=3$) and polyhedral ($m=6$) estimators of the log of drag coefficient versus log of Reynolds number and lift coefficient for a fixed thickness ratio $\tau = 8\%$.*

transformation, the data set can be approximated well by a convex function. We fit both spectrahedral and polyhedral functions to this data set, and the best fits for the whole data set appear in Figure 4 for models with 6 degrees for freedom per dimension. Then, we performed hold-out validation, training on 80% of the data, and testing on the remaining 20%. The RMSE is given in Table 3, where we observe that the spectrahedral estimator achieves a smaller error than polyhedral regression.

4.3.2. Power modeling for circuit design. A circuit is an interconnected collection of electrical components including batteries, resistors, inductors, capacitors, logical gates, and transistors. In circuit design, the goal is to optimize over variables such as devices, gates, threshold, and power supply voltages in order to minimize circuit delay or physical area. The power dissipated, P, is a function of gate supply V_{dd} and threshold voltages V_{th}. The following model (see [14] and [12]) can be used to study this relationship:

$$P = V_{dd}^2 + 30V_{dd}e^{-(V_{th}-0.06V_{dd})/0.039}.$$

We generate n i.i.d. data points as in [12] as follows. For each input-output pair, first sample $u=(V_{dd}, V_{th})$ uniformly over the domain $1.0 \le V_{dd} \le 2.0$ and $0.2 \le V_{th} \le 0.4$ and compute $P(u)$. Then, apply the transformation $(x, y) = (\log u, \log P(u))$. We fit this collection of transformed data points using polyhedral and spectrahedral regression, and the estimators for $n = 20$, 50, and 200 are illustrated in Figure 5. We also perform hold-out validation with 20% of the data for the case $n = 200$ and the RMSE appears in in Table 3. By this measure, the spectrahedral estimator performs much better than the polyhedral estimator in this application.

5. Discussion and future work. In this work, we have introduced spectrahedral regression as a new method for estimating a convex function from noisy measurements. Spectrahedral estimators are appealing from a qualitative and quantitative perspective, and we have shown they hold advantages over the usual LSE methods as well as polyhedral estimators when the underlying convex function is nonpolyhedral. Our theoretical results and numerical experiments call for further study of the expressivity of this model class and its computational advantages. We now describe a few directions of future research.

5.1. Parameter selection and tuning. In our proposed method, the model parameters m and k must be chosen in advance to obtain a spectrahedral estimator.

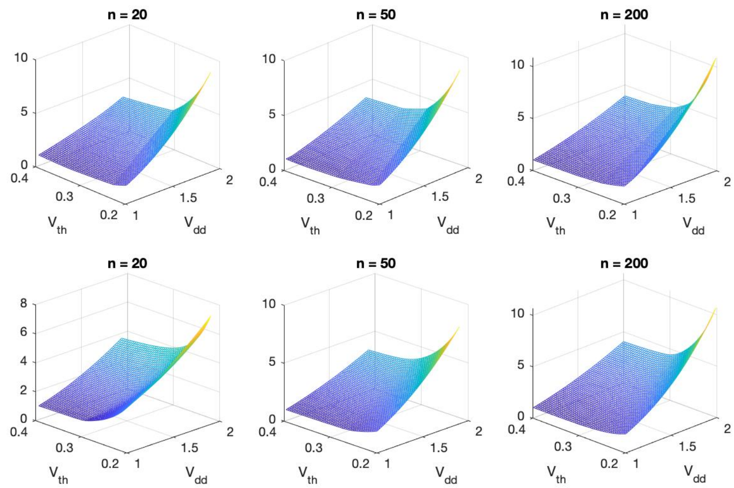

FIG. 5. *Polyhedral ($m = 6$) and spectrahedral ($m = 3$) estimators of $n = 20$, 50, and 200 transformed data points generated from the power dissipation model.*

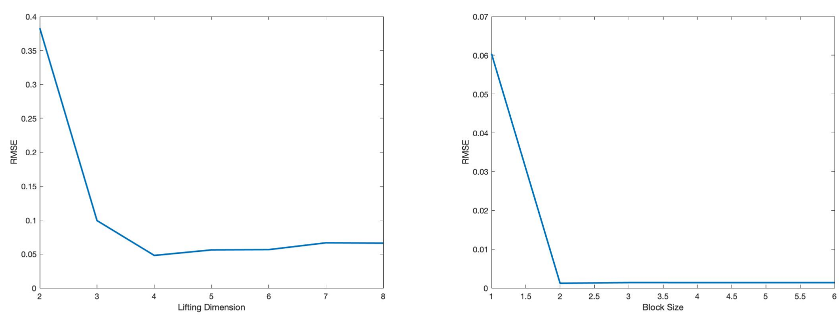

FIG. 6. *Data-driven tuning of parameters m and k. The left plot shows hold-out validation error for spectrahedral estimators with varying parameter m obtained from noisy measurements of an $m = 3$ spectrahedral function. The right plot shows hold-out validation error for $m = 6$ spectrahedral estimators of varying block size k obtained from noisy measurements of a $k = 2$ spectrahedral function.*

For small m and k, the estimator is efficient to compute, and the resulting estimator has a more compact description but may underfit the data. It would be very useful to develop adaptive methods for choosing these parameters using the data set. Here we describe an experiment to choose m and k using hold-out validation. Figure 6 shows two plots illustrating two experiments. The first experiment uses hold-out validation to find an appropriate m, and the second experiment finds an appropriate k given a fixed m. Spectrahedral estimators were obtained using a test data set of size 200 for varying m and k, and the RMSE from a test data set of size 200 was computed for each. In the first experiment, the test and training data sets are generated from a random spectrahedral function with $k = m = 3$. We see from the plot that $m = 3$ would indeed be the appropriate choice to model this data set. In the second experiment, the data was generated from a random spectrahedral function with $k = 2$ and $m = 6$. If we initially chose $m = 6$, the plot shows that $k = 2$ would indeed be the best choice for the block sizes for the model.

5.2. Expressiveness of spectrahedral functions. An interesting open question is to obtain the approximation rate for (m,k)-spectrahedral functions to the class of Lipschitz convex functions and (m,k)-spectratopes to the class of convex bodies for general k. There is extensive literature on this approximation question for polytopes (see, for instance, [3, 6]), and we have obtained matching bounds (up to logarithmic factors) for fixed $k>1$. For k depending on m, and in particular in the case $k=m$, the literature is more limited; one example is [2]. Progress in this direction would complete our understanding of the expressive power of the model presented here and have important consequences for how well semidefinite programming can approximate a general convex program.

5.3. Computational guarantees. We have also proved computational guarantees for a natural alternating minimization algorithm for spectrahedral regression. However, this convergence guarantee depends on a good initialization. In practice, running the algorithm with multiple random initializations and taking the estimator with the smallest error works well, but it would be very interesting to extend the results on initialization in [8] to the spectrahedral case. Another line of future work is to extend other methods to solve the nonconvex optimization (1.3) in the polyhedral case such as the adaptive partitioning method in [13] and the method proposed in [28]. These algorithms also lack theoretical guarantees, and it would be interesting to obtain conditions under which these methods obtain good estimates of the true parameter.

Appendix A. Lemmas for the proofs of Theorems 3.1 and 3.2. We first give a few definitions that are needed in following lemmas. A random vector $\xi\in\mathbb{R}^d$ is sub-Gaussian with parameter η if $\mathbb{E}[\xi]=0$ and for each $u\in S^{d-1}$, $\mathbb{E}[e^{\lambda\langle u,\xi\rangle}]\le e^{\lambda^2\eta^2/2}$ for all $\lambda\in\mathbb{R}$. The sub-Gaussian norm of a random variable X, denoted by $\|X\|_{\psi_2}$, is defined as

$$\|X\|_{\psi_2}=\inf\{t>0:\mathbb{E}[\exp(X^2/t^2)]\le 2\}.$$

For $\xi\in\mathbb{R}^d$, the sub-Gaussian norm is defined as $\|\xi\|_{\psi_2}:=\sup_{u\in\mathbb{S}^{d-1}}\|\langle\xi,u\rangle\|_{\psi_2}$. The subexponential norm of a random variable X, denoted by $\|X\|_{\psi_1}$, is defined as

$$\|X\|_{\psi_1}=\inf\{t>0:\mathbb{E}[\exp(|X|/t)]\le 2\},$$

and the subexponential norm of a random vector ξ is defined similarly.

We also recall that the covering number of a Euclidean ball satisfies

$$\mathcal{N}(B_q(z,R),\|\cdot\|_2,\varepsilon)\le(1+2R/\varepsilon)^q \tag{A.1}$$

for $\varepsilon\le 2R$ by a standard volume argument.

The proofs rely on uniform spectral concentration bounds of a sample covariance matrix, which follow from Bernstein's inequality and Dudley's inequality. A general reference for the ideas in the lemmas below is [32].

LEMMA A.1. *Let ξ be an η-sub-Gaussian random vector in $\mathbb{R}^d$ such that $\mathbb{E}[\xi_i^2]=1$ and, let U be any $m\times m$ matrix with $\|U\|_F=1$. Let $\Sigma:=\mathbb{E}[(\xi\otimes U)(\xi\otimes U)^T]\in\mathbb{S}^{dm^2}$. Then, the following inequalities hold:*

(i) $\|\xi\otimes U\|_{\psi_2}\le d^{1/2}\eta$;
(ii) $m^{-1}\le\|\Sigma\|_{op}\le d^{1/2}$;
(iii) $\|\xi\otimes U\|_{\psi_2}\le(md)^{1/2}\eta\|\Sigma\|_{op}^{1/2}$.

Proof. Recall that $\xi \otimes U$ is sub-Gaussian if $\langle \xi \otimes U, v\rangle$ is sub-Gaussian for every $v \in S^{dm^2-1}$. Indeed, we first see that

$$\|\xi \otimes U\|_F^2 = \|\xi\|_2^2 \|U\|_F^2 = \|\xi\|_2^2.$$

Then, by Lemma 2.7.6 in [32] and the triangle inequality

$$\|\langle \xi \otimes U, v\rangle\|_{\psi_2}^2 \le \big\| \|\xi\|^2 \big\|_{\psi_1} = \left\| \sum_{i=1}^d \xi_i^2 \right\|_{\psi_1} \le \sum_{i=1}^d \|\xi_i\|_{\psi_2}^2 \le d\eta^2.$$

For the second claim, we see that

$$\|\Sigma\|_{op}^2 \ge \frac{1}{dm^2}\|\Sigma\|_F^2 = \frac{1}{dm^2}\mathbb{E}\left[\sum_{i=1}^d \sum_{j,k=1}^m \xi_i^2 U_{ik}^2\right] = \frac{1}{m^2},$$

and

$$\|\Sigma\|_{op}^2 \le \|\Sigma\|_F^2 = \mathbb{E}\left[\sum_{i=1}^d \sum_{j,k=1}^m \xi_i^2 U_{ik}^2\right] = d.$$

This implies the final claim $\|\langle \xi \otimes U, v\rangle\|_{\psi_2} \le \eta d^{1/2} \le \eta (md)^{1/2} \|\Sigma\|_{op}^{1/2}$. ∎

For the next lemmas, recall that for a random vector $\xi \in \mathbb{R}^d$, for each $\mathcal{A} \in (\mathbb{S}_k^m)^d$ we define $U_{\mathcal{A}}$ to be the rank one matrix such that

$$\langle \xi \otimes U_{\mathcal{A}}, \mathcal{A}\rangle = \langle U_{\mathcal{A}}, \mathcal{A}[\xi]\rangle = \lambda_{\max}(\mathcal{A}[\xi]).$$

Also, define for $r > 0$ the set

$$B(\mathcal{A}_*, r) := \{\mathcal{A} \in (\mathbb{S}_k^m)^d : \|\mathcal{A} - \mathcal{A}_*\|_F \le r\}.$$

LEMMA A.2. *Consider the setting of Theorem* 3.1. *Then, for all* $\mathcal{A}_1, \mathcal{A}_2 \in B(\mathcal{A}_*, \kappa/4)$,

$$\|U_{\mathcal{A}_1} - U_{\mathcal{A}_2}\|_F^2 \le \frac{32\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2}{\kappa^2},$$

and

$$\|\xi \otimes U_{\mathcal{A}_1} - \xi \otimes U_{\mathcal{A}_2}\|_{\psi_2} \le \frac{2^{5/2}\tilde{\eta}((d+1)m)^{1/2}}{\kappa}\left(\inf_{\mathcal{A} \in B(\mathcal{A}_*, \kappa/4)} \|\Sigma_{\mathcal{A}}^{1/2}\|_{op}\right) \|\mathcal{A}_1 - \mathcal{A}_2\|_F.$$

Proof. First note that for all $\mathcal{A} \in B(\mathcal{A}_*, \kappa/4)$, Weyl's inequality implies that for all $u \in S^{d-1}$,

$$\lambda_1(\mathcal{A}[u]) - \lambda_2(\mathcal{A}[u]) \ge \lambda_1(\mathcal{A}_*[u]) - \lambda_2(\mathcal{A}_*[u]) - 2\|\mathcal{A} - \mathcal{A}_*\|_{op} \ge \frac{\kappa}{2} > 0.$$

Then, observe that $\|U_{\mathcal{A}_1} - U_{\mathcal{A}_2}\|_F^2 = 2\sin(\Theta(u_1, u_2))^2$, where $U_{\mathcal{A}_1} = u_1 u_1^T$, $U_{\mathcal{A}_2} = u_2 u_2^T$, and $u_1, u_2 \in S^{m-1}$. By a variation of the Davis–Kahan theorem (Theorem 2 in [33]),

$$\sin(\Theta(u_1, u_2))^2 \le \frac{4\|(\mathcal{A}_1 - \mathcal{A}_2)[\xi]\|_{op}^2}{(\lambda_1(\mathcal{A}_2[\xi]) - \lambda_2(\mathcal{A}_2[\xi]))^2} \le \frac{16\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2}{\kappa^2}.$$

This implies that

$$\|U_{\mathcal{A}_1} - U_{\mathcal{A}_2}\|_F^2 \leq \frac{32}{\kappa^2}\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2,$$

and by Lemma A.1,

$$\begin{aligned}\|\langle \xi \otimes (U_{\mathcal{A}_1} - U_{\mathcal{A}_2}), v\rangle\|_{\psi_2}^2 &\leq \frac{32\tilde{\eta}^2(d+1)}{\kappa^2}\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2 \\ &\leq \frac{32\tilde{\eta}^2(d+1)m}{\kappa^2}\left(\inf_{\mathcal{A}\in B(\mathcal{A}_*,\kappa/4)}\|\Sigma_{\mathcal{A}}\|_{op}\right)\|\mathcal{A}_1 - \mathcal{A}_2\|_F^2. \qquad \square\end{aligned}$$

LEMMA A.3. *Define* $B_q(z,R) := \{x \in \mathbb{R}^q : \|x - z\|_2 \leq R\}$ *for* $R > 0$ *and* $z \in \mathbb{R}^q$. *Let* $\{\xi_a\}_{a\in B_q(z,R)}$ *be a stochastic process in* $\mathbb{R}^d$ *such that*

(i) $\|\xi_a\|_{\psi_2} \leq \eta\|\Sigma_a^{1/2}\|_{op}$ *for some* $\eta \geq 1$;

(ii) *for all* a_1, $a_2 \in B_q(z,R)$, $\|\xi_{a_1} - \xi_{a_2}\|_{\psi_2} \leq K(\inf_{a\in B_q(z,R)}\|\Sigma_a^{1/2}\|_{op})\|a_1 - a_2\|_2$, *where* $\Sigma_a = \mathbb{E}[\xi_a\xi_a^T]$ *for all* $a \in B_q(z,R)$. *Define* $\Xi_a \in \mathbb{R}^{n\times d}$ *to be the matrix with* n *i.i.d. rows in* $\mathbb{R}^d$ *distributed as* ξ_a, *and let* $\Sigma_a := \mathbb{E}[\xi_a\xi_a^T]$. *Fix* $\tau > 0$. *Then, there exist absolute constants* $c_0, c_1 > 0$ *such that if* $n \geq c_0\tau^{-2}K^2\eta^4R^2\max\{q,d\}$,

$$\mathbb{P}\left(\sup_{a\in B_q(z,R)}\|\Sigma_a\|_{op}^{-1}\left\|\frac{1}{n}\Xi_a^T\Xi_a - \Sigma_a\right\|_{op} \geq \tau\right) \leq 2e^{-c_1n\tau^2/K^2\eta^4R^2}.$$

This implies that with probability greater than $1 - 2e^{-c_1n\tau^2/K^2\eta^4R^2}$,

$$1 - \tau \leq \inf_{a\in B_q(z,R)}\frac{\lambda_{\min}\left(\Xi_a^T\Xi_a\right)}{n\lambda_{\max}(\Sigma_a)} \leq \sup_{a\in B_q(z,R)}\frac{\lambda_{\max}\left(\Xi_a^T\Xi_a\right)}{n\lambda_{\max}(\Sigma_a)} \leq 1 + \tau.$$

Proof. First suppose that ξ_a is isotropic for all a, i.e., $\Sigma_a = I$. For the general case, the conclusion follows from the inequality

$$\left\|\frac{1}{n}\Xi_a^T\Xi_a - \Sigma_a\right\|_{op} \leq \|\Sigma_a\|_{op}\left\|\frac{1}{n}\sum_{i=1}^n(\Sigma_a^{-1/2}\xi_a^{(i)})(\Sigma_a^{-1/2}\xi_a^{(i)})^T - I\right\|_{op}.$$

We first show that for any $x \in \mathbb{S}^{d-1}$, the stochastic process $X_a := \frac{1}{\sqrt{n}}\|\Xi_a x\| - 1$ has sub-Gaussian increments $\|X_{a_1} - X_{a_2}\|_{\psi_2} = \frac{1}{\sqrt{n}}\|\|\Xi_{a_1}x\|_2 - \|\Xi_{a_2}x\|_2\|_{\psi_2}$.

Case 1: $s \in [0, 4K\sqrt{n}]$. We first see that

$$\begin{aligned}&\mathbb{P}\left(|\|\Xi_{a_1}x\|_2 - \|\Xi_{a_2}x\|_2| \geq s\|a_1 - a_2\|_2\right) \\ &\quad = \mathbb{P}\left(\frac{\left|\|\Xi_{a_1}x\|_2^2 - \|\Xi_{a_2}x\|_2^2\right|}{\|a_1 - a_2\|} \geq s(\|\Xi_{a_1}x\|_2 + \|\Xi_{a_2}x\|_2)\right) \\ &\quad \leq \mathbb{P}\left(\frac{\left|\|\Xi_{a_1}x\|_2^2 - \|\Xi_{a_2}x\|_2^2\right|}{\|a_1 - a_2\|} \geq s\|\Xi_{a_1}x\|_2\right) \\ &\quad \leq \mathbb{P}\left(\frac{\left|\|\Xi_{a_1}x\|_2^2 - \|\Xi_{a_2}x\|_2^2\right|}{\|a_1 - a_2\|} \geq \frac{s\sqrt{n}}{2}\right) + \mathbb{P}\left(\|\Xi_{a_1}x\|_2 \leq \frac{\sqrt{n}}{2}\right) \\ &\quad \leq \mathbb{P}\left(\frac{\left|\|\Xi_{a_1}x\|_2^2 - \|\Xi_{a_2}x\|_2^2\right|}{\|a_1 - a_2\|} \geq \frac{s\sqrt{n}}{2}\right) + \mathbb{P}\left(\left|\|\Xi_{a_1}x\|_2 - \sqrt{n}\right| \geq \frac{s}{8K}\right).\end{aligned} \tag{A.2}$$

Then, note that

$$\|\Xi_{a_1}x\|_2^2 - \|\Xi_{a_2}x\|_2^2 = \sum_{i=1}^{n} \langle \xi_{a_1}^{(i)} - \xi_{a_2}^{(i)}, x\rangle \langle \xi_{a_1}^{(i)} + \xi_{a_2}^{(i)}, x\rangle,$$

and by Lemma 2.7.7 in [32],

$$\begin{aligned}\|\langle \xi_{a_1}^{(i)} - \xi_{a_2}^{(i)}, x\rangle \langle \xi_{a_1}^{(i)} + \xi_{a_2}^{(i)}, x\rangle\|_{\psi_1} &\leq \|\langle \xi_{a_1}^{(i)} - \xi_{a_2}^{(i)}, x\rangle\|_{\psi_2} \|\langle \xi_{a_1}^{(i)} + \xi_{a_2}^{(i)}, x\rangle\|_{\psi_2} \\ &\leq 2\eta K \|a_1 - a_2\|_2.\end{aligned}$$

Each term in the sum also has zero mean. Indeed,

$$\mathbb{E}[\langle \xi_{a_1}^{(i)} - \xi_{a_2}^{(i)}, x\rangle \langle \xi_{a_1}^{(i)} + \xi_{a_2}^{(i)}, x\rangle] = \mathbb{E}[\langle \xi_{a_1}^{(i)}, x\rangle^2 - \langle \xi_{a_2}^{(i)}, x\rangle^2] = 0.$$

Applying Bernstein's inequality (Corollary 2.8.3 in [32]) gives, for all $t \geq 0$,

$$\mathbb{P}\left(\frac{\|\Xi_{a_1}x\|_2^2 - \|\Xi_{a_2}x\|_2^2}{\|a_1 - a_2\|} \geq t\right) \leq 2e^{-c_1 \min\left\{\frac{t^2}{4\eta^2 K^2 n}, \frac{t}{2\eta K}\right\}}.$$

For the second tail probability in (A.2), Theorem 3.1.1 in [32] implies

$$\mathbb{P}\left(\left|\|\Xi_{a_1}x\|_2 - \sqrt{n}\right| \geq t\right) \leq 2e^{-\frac{c_2 t^2}{\eta^4}},$$

where we have used that ξ_a is isotropic. Thus, since $s < 4K\sqrt{n}$ and $\eta \geq 1$,

(A.3)
$$\mathbb{P}\left(\frac{|\|\Xi_{a_1}x\|_2 - \|\Xi_{a_2}x\|_2|}{\|a_1 - a_2\|_2} \geq s\right) \leq 2e^{-c_1 \min\{\frac{s^2}{16\eta^2 K^2}, \frac{s\sqrt{n}}{4\eta K}\}} + 2e^{-\frac{c_2 s^2}{64\eta^4 K^2}} \leq 4e^{-\frac{c_3 s^2}{\eta^4 K^2}}.$$

Case 2: $s \geq 4K\sqrt{n}$. By the triangle inequality,

$$\begin{aligned}&\mathbb{P}\left(\frac{|\|\Xi_{a_1}x\|_2 - \|\Xi_{a_2}x\|_2|}{\|a_1 - a_2\|} \geq s\right) \leq \mathbb{P}\left(\frac{\|(\Xi_{a_1} - \Xi_{a_2})x\|^2}{\|a_1 - a_2\|^2} \geq s^2\right) \\ &= \mathbb{P}\left(\frac{\|(\Xi_{a_1} - \Xi_{a_2})x\|^2}{\|a_1 - a_2\|^2} - n\frac{\mathbb{E}[\langle \xi_{a_1} - \xi_{a_2}, x\rangle^2]}{\|a_1 - a_2\|^2} \geq s^2 - n\frac{\mathbb{E}[\langle \xi_{a_1} - \xi_{a_2}, x\rangle^2]}{\|a_1 - a_2\|^2}\right) \\ &\leq \mathbb{P}\left(\frac{\|(\Xi_{a_1} - \Xi_{a_2})x\|^2}{\|a_1 - a_2\|^2} - n\frac{\mathbb{E}[\langle \xi_{a_1} - \xi_{a_2}, x\rangle^2]}{\|a_1 - a_2\|^2} \geq s^2 - 4K^2 n\right) \\ &\leq \mathbb{P}\left(\left|\frac{\|(\Xi_{a_1} - \Xi_{a_2})x\|^2}{\|a_1 - a_2\|^2} - n\frac{\mathbb{E}[\langle \xi_{a_1} - \xi_{a_2}, x\rangle^2]}{\|a_1 - a_2\|^2}\right| \geq \frac{3s^2}{4}\right).\end{aligned}$$

where for the second to last inequality we have used that

$$\mathbb{E}[\langle \xi_{a_1} - \xi_{a_2}, x\rangle^2] \leq 4\|\xi_{a_1} - \xi_{a_2}\|_{\psi_2}^2 \leq 4K^2 \|a_1 - a_2\|_2^2,$$

and the last inequality follows from the lower bound on s and the fact that $\eta \geq 1$. By Bernstein's inequality again (Corollary 2.8.3 in [32]) and the lower bound on s,

$$\mathbb{P}\left(\frac{|\|\Xi_{a_1}x\|_2 - \|\Xi_{a_2}x\|_2|}{\|a_1 - a_2\|} \geq s\right) \leq 2e^{-c_4 \min\left\{\frac{s^4}{nK^4}, \frac{s^2}{K^2}\right\}} \leq 2e^{-\frac{c_4 s^2}{K^2}}. \tag{A.4}$$

Combining (A.3) and (A.4) with Proposition 2.5.2 in [32] then implies

$$\|X_{a_1} - X_{a_2}\|_{\psi_2} \leq \frac{K\eta^2}{\sqrt{n}} \|a_1 - a_2\|_2,$$

where we have used that $\eta \geq$.

By Theorem 8.1.6 in [32] and (A.1), we have, for $\delta > 0$ and $n \geq q\delta^{-2}$,

$$\sup_{a \in B_q(z,R)} \left| \frac{1}{\sqrt{n}} \|\Xi_a x\| - 1 \right| \leq \frac{c_5 K \eta^2}{\sqrt{n}} \left(c_6 R\sqrt{q} + 2R\delta\sqrt{n} \right) \leq c_7 K \eta^2 R \delta \tag{A.5}$$

with probability greater than $1 - 2e^{-\delta^2 n}$. Now, let $\tau > 0$. By the inequality $|z^2 - 1| \leq 3\max\{|z-1|, |z-1|^2\}$, for all $z \geq 0$,

$$\mathbb{P}\left(\sup_{a \in B_q(z,R)} \left| \frac{1}{n} \|\Xi_a x\|_2^2 - 1 \right| \geq \frac{\tau}{2} \right) \leq \mathbb{P}\left(\sup_{a \in B_q(z,R)} \left| \frac{1}{\sqrt{n}} \|\Xi_a x\|_2 - 1 \right| \geq \frac{\tau}{6} \right).$$

Letting $\delta = \frac{\tau}{6c_7 K \eta^2 R}$ in (A.5) gives the following. For $n \geq c_8 q K^2 \eta^4 R^2 \tau^{-2}$,

$$\mathbb{P}\left(\sup_{a \in B_q(z,R)} \left| \frac{1}{n} \|\Xi_a x\|_2^2 - 1 \right| \geq \frac{\tau}{2} \right) \leq 2e^{-n\tau^2 / c_8 K^2 \eta^4 R^2}.$$

Finally, by Lemma 5.3 in [31],

$$\sup_{a \in B_q(r)} \left\| \frac{1}{n} \Xi_a^T \Xi_a - I \right\|_{op} \leq 2 \max_{x \in \mathcal{N}} \sup_{a \in B_q(r)} \left| \frac{1}{n} \|\Xi_a x\|_2^2 - 1 \right|,$$

where $\mathcal{N}$ is a $\frac{1}{4}$-net of the unit sphere S^{d-1}. Lemma 5.4 in [31] implies $|\mathcal{N}| \leq 9^d$. Applying the union bound then gives, for $n \geq c_8 q K^2 \eta^4 R^2 \tau^{-2}$,

$$\begin{aligned} \mathbb{P}\left(\sup_{a \in B_q(z,R)} \left\| \frac{1}{n} \Xi_a^T \Xi_a - I \right\|_{op} \geq \tau \right) &\leq \mathbb{P}\left(\max_{x \in \mathcal{N}} \sup_{a \in B_q(z,R)} \left| \frac{1}{n} \|\Xi_a x\|_2^2 - \|x\|_2^2 \right| \geq \frac{\tau}{2} \right) \\ &\leq |\mathcal{N}| \mathbb{P}\left(\sup_{a \in B_q(z,R)} \left| \frac{1}{n} \|\Xi_a x\|_2^2 - 1 \right| \geq \frac{\tau}{2} \right) \leq 2 \cdot 9^d e^{-n\tau^2 / c_9 K^2 \eta^4 R^2}. \end{aligned}$$

Thus, there exist absolute constants b_1, b_2 such that for $n \geq b_1 \tau^{-2} K^2 \eta^4 R^2 \max\{q, d\}$,

$$\mathbb{P}\left(\sup_{a \in B_q(z,R)} \left\| \frac{1}{n} \Xi_a^T \Xi_a - I \right\|_{op} \geq \tau \right) \leq 2e^{-b_2 n \tau^2 / K^2 \eta^4 R^2}.$$ ∎

LEMMA A.4. *Consider the setting of Theorem* 3.1. *Let* $B_* := B(\mathcal{A}_*, \kappa/4)$, *and define the class of orthogonal projections* $\mathcal{P} := \{P_{\Xi_{\mathcal{A}}} : \mathcal{A} \in B_*\}$. *Then, there exist absolute constants* c_i, $i = 0, \ldots, 3$, *such that for* $n \geq c_0 \max\{\tilde{\eta}^6 m^4 (d+1), \tilde{\eta}^2 m^3 \max\{1, \sigma^2\}\}$,

$$\mathbb{P}\left(\sup_{P \in \mathcal{P}} \|P\varepsilon\|_2^2 \geq c_1 \sigma^2 \log(n)^2 (d+1) m^2 \right) \leq n^{-c_2 m} + 2e^{-c_3 n / (\tilde{\eta}^6 (d+1) m^4)}.$$

Proof. We first observe that for $P = P_{\Xi_{\mathcal{A}}} \in \mathcal{P}$,

$$\|P\varepsilon\|_2^2 = \|\Xi_{\mathcal{A}} (\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}})^{-1} \Xi_{\mathcal{A}}^T \varepsilon\|_2^2 \leq \|(\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}})^{-1}\|_{op} \|\Xi_{\mathcal{A}}^T \varepsilon\|_2^2 = \frac{\|\Xi_{\mathcal{A}}^T \varepsilon\|_2^2}{\lambda_{\min}(\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}})} \leq \frac{\|\Xi_{\mathcal{A}}^T \varepsilon\|_2^2}{\lambda_{\min}(\Xi_{\mathcal{A}}^T \Xi_{\mathcal{A}})}.$$

Then, for $t > 0$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\frac{\|\Xi_{\mathcal{A}}^T\varepsilon\|_2^2}{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}\geq t\right) &\leq \mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\frac{\|\Xi_{\mathcal{A}}^T\varepsilon\|_2^2}{n\|\Sigma_{\mathcal{A}}\|_{op}}\geq t\inf_{\mathcal{A}}\frac{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}{n\|\Sigma_{\mathcal{A}}\|_{op}}\right)\\
&= \mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\frac{\|\Xi_{\mathcal{A}}^T\varepsilon\|_2^2}{n\|\Sigma_{\mathcal{A}}\|_{op}}\geq \frac{t}{2}\right)+\mathbb{P}\left(\inf_{\mathcal{A}\in B_*}\frac{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}{n\|\Sigma_{\mathcal{A}}\|_{op}}\leq\frac{1}{2}\right)\\
&\leq \mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|_2^2\geq\frac{tn}{2m}\right)+\mathbb{P}\left(\inf_{\mathcal{A}\in B_*}\frac{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}{n\|\Sigma_{\mathcal{A}}\|_{op}}\leq\frac{1}{2}\right), \qquad \text{(A.6)}
\end{aligned}$$

where the last inequality follows from Lemma A.1. To upper bound the second probability above, Lemmas A.2 and A.3 imply that for $n \geq c_0\tilde{\eta}^6(d+1)m^4$

$$\mathbb{P}\left(\inf_{\mathcal{A}\in B(\mathcal{A}_*,\kappa/4)}\frac{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}{n\|\Sigma_{\mathcal{A}}\|_{op}}\leq 1-\frac{1}{2}\right)\leq 2\exp\{-c_1 n/(\tilde{\eta}^6(d+1)m^4)\}. \qquad \text{(A.7)}$$

We now turn to the first probability in (A.6). First note that for all $\mathcal{A}\in B_*$,

(A.8)
$$\mathbb{E}[\|\Xi_{\mathcal{A}}\varepsilon\|^2] = \mathbb{E}[\|\Xi_{\mathcal{A}}\|_F^2]\sigma^2 = \sum_{i=1}^n \mathbb{E}[\|\xi^{(i)}\otimes U_{\mathcal{A}}^{(i)}\|^2]\sigma^2 = \sum_{i=1}^n \mathbb{E}[\|\xi^{(i)}\|^2\|U_{\mathcal{A}}^{(i)}\|^2]\sigma^2 = n(d+1)\sigma^2.$$

In particular, the expectation does not depend on $\mathcal{A}$. Then,

$$\begin{aligned}
&\sup_{\mathcal{A}\in B_*}\left(\|\Xi_{\mathcal{A}}\varepsilon\|^2-\mathbb{E}[\|\Xi_{\mathcal{A}}\varepsilon\|^2]\right)-\mathbb{E}\sup_{\mathcal{A}\in B_*}\left(\|\Xi_{\mathcal{A}}\varepsilon\|^2-\mathbb{E}[\|\Xi_{\mathcal{A}}\varepsilon\|^2]\right)\\
&\qquad = \sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}\varepsilon\|^2-\mathbb{E}\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}\varepsilon\|^2.
\end{aligned}$$

Now, recall that $M := \|\max_{i=1,\dots,n}\varepsilon_i\|_{\psi_2}\leq c_0\sigma\sqrt{\log n}$ for an absolute constant c_0 [18]. Applying Theorem 1.1 in [15] to the family of matrices $\{\Xi_{\mathcal{A}}\Xi_{\mathcal{A}}^T : \mathcal{A}\in B_*\}$ gives the following.

For $s\geq\max\{c_2\sigma\sqrt{\log(n)}\mathbb{E}[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|_2], c_2^2\sigma^2\log(n)n(d+1)\}$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2-\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2\right]\geq s\right)\\
&\leq \exp\left(-\frac{c_2}{\sigma^2\log(n)}\min\left\{\frac{s^2}{\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|\right]^2},\frac{s}{\mathbb{E}[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}\|_{op}^2]}\right\}\right).
\end{aligned}$$

Also by (A.8), $\mathbb{E}[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2]\geq n(d+1)\sigma^2$, and thus, for s as above,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2-n(d+1)\sigma^2\geq s\right)\\
&\leq \exp\left(-\frac{c_2}{\sigma^2\log(n)}\min\left\{\frac{s^2}{\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|\right]^2},\frac{s}{n(d+1)}\right\}\right). \qquad \text{(A.9)}
\end{aligned}$$

We now upper bound $\mathbb{E}[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|]$, so we first define the stochastic process $X_{\mathcal{A}} := \|\Xi_{\mathcal{A}}^T\varepsilon\|$. For $\mathcal{A}$ and $\mathcal{B}$ in $(\mathbb{S}_k^m)^d$,

$$\|\Xi_{\mathcal{A}}-\Xi_{\mathcal{B}}\|_F^2 = \sum_{i=1}^n\|\xi^{(i)}\otimes(U_{\mathcal{A}}^{(i)}-U_{\mathcal{B}}^{(i)})\|_F^2 = \sum_{i=1}^n\|\xi^{(i)}\|_2^2\|U_{\mathcal{A}}^{(i)}-U_{\mathcal{B}}^{(i)}\|_F^2.$$

By the assumptions of Theorem 3.1 and Lemma A.2, for all $\mathcal{A},\mathcal{B}\in B_*$,

$$\|\|\Xi_{\mathcal{A}}-\Xi_{\mathcal{B}}\|_F\|_{\psi_2}^2=\|\|\Xi_{\mathcal{A}}-\Xi_{\mathcal{B}}\|_F^2\|_{\psi_1}\le\sum_{i=1}^n\|\|\xi^{(i)}\|_2^2\|U_{\mathcal{A}}^{(i)}-U_{\mathcal{B}}^{(i)}\|_F^2\|_{\psi_1}\le\frac{c_3n\tilde\eta^2}{\kappa^2}\|\mathcal{A}-\mathcal{B}\|_F^2.$$

Then, by Lemma 2.7.5 in [32] and the Hanson–Wright inequality [23, Theorem 2.1], since ε is independent of $\{\Xi_{\mathcal{A}}\}_{\mathcal{A}\in B_*}$, there is a constant c_4 such that

$$\begin{aligned}\|X_{\mathcal{A}}-X_{\mathcal{B}}\|_{\psi_1}^2&\le\left\|\frac{\|\Xi_{\mathcal{A}}^T\varepsilon\|_2-\|\Xi_{\mathcal{B}}^T\varepsilon\|_2}{\|\Xi_{\mathcal{A}}-\Xi_{\mathcal{B}}\|_F}\right\|_{\psi_2}^2\|\|\Xi_{\mathcal{A}}-\Xi_{\mathcal{B}}\|_F\|_{\psi_2}^2\\&\le\left\|\frac{\|(\Xi_{\mathcal{A}}^T-\Xi_{\mathcal{B}}^T)\varepsilon\|_2}{\|\Xi_{\mathcal{A}}-\Xi_{\mathcal{B}}\|_F}\right\|_{\psi_2}^2\|\|\Xi_{\mathcal{A}}-\Xi_{\mathcal{B}}\|_F\|_{\psi_2}^2\le\frac{c_4n\sigma^4\tilde\eta^2}{\kappa^2}\|\mathcal{A}-\mathcal{B}\|_F^2.\end{aligned}$$

Thus, $\{X_{\mathcal{A}}\}_{\mathcal{A}\in B_*}$ has subexponential increments, and by Theorem 2.2.4 in [30] (with $\psi(x)=e^x-1$) and (A.1),

$$\begin{aligned}\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|\right]&\le\mathbb{E}[\|\Xi_{\mathcal{A}_*}^T\varepsilon\|]+\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|-\|\Xi_{\mathcal{A}_*}^T\varepsilon\|\right]\\&\le c_5\left(\sigma\sqrt{n(d+1)}+\frac{\sqrt{n}\sigma^2\tilde\eta m^2(d+1)}{\kappa}\int_0^{\kappa/4}\log\left(\frac{2\kappa}{\varepsilon}\right)\mathrm{d}\varepsilon\right)\\&\le c_6\max\{\sigma,\sigma^2\}\tilde\eta(d+1)m^2\sqrt{n}.\end{aligned}$$

Then, for $s\ge\max\{c_7\sqrt{n\log(n)}\max\{\sigma^2,\sigma^3\}\tilde\eta(d+1)m^2,c_0^2\sigma^2n\log(n)(d+1)\}$,

$$\begin{aligned}&\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2-n(d+1)\sigma^2\ge s\right)\\&\le\exp\left(-\frac{c_8}{\sigma^2\log(n)}\min\left\{\frac{s^2}{\max\{\sigma^2,\sigma^4\}\tilde\eta^2(d+1)^2m^4n},\frac{s}{n(d+1)}\right\}\right).\end{aligned}$$

Letting $t=c_9\sigma^2\log(n)^2(d+1)m^2$ for a constant $c_9>0$ large enough,

$$\begin{aligned}&\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2\ge\frac{tn}{2m}\right)=\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2\ge c_{10}\sigma^2n\log(n)^2(d+1)m\right)\\&\le\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2-n(d+1)\sigma^2\ge c_{11}\sigma^2n\log(n)^2(d+1)m\right)\\&\le\exp\left(-c_{12}\min\left\{\frac{n\log(n)^3}{\max\{1,\sigma^2\}\tilde\eta^2m^2},m\log(n)\right\}\right)\le n^{-c_{12}m}\end{aligned}$$

for $n\ge m^3\max\{1,\sigma^2\}\tilde\eta^2$. Finally, combining the above bound with (A.7) and (A.6) gives

$$\mathbb{P}\left(\sup_{P\in\mathcal{P}}\|P\varepsilon\|_2^2\ge c_4\sigma^2\log(n)^2(d+1)m^2\right)\le n^{-c_{12}m}+2e^{-c_1n/(\tilde\eta^6(d+1)m^4)}.\qquad\square$$

LEMMA A.5. *Consider the setting of Theorem* 3.2 *and let* $B_*:=B(\mathcal{A}_*,\frac{k\kappa}{4m\tilde\eta\sqrt{d+1}})$. *For all* $\mathcal{A}_1,\mathcal{A}_2\in B_*$,

$$\mathbb{E}[\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F]\le\frac{16m^{3/2}\max\{1,c\}\tilde\eta\sqrt{d+1}}{k^{3/2}\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F,\tag{A.10}$$

and

(A.11)
$$\mathbb{P}\left(\bigcup_{\mathcal{A}_1,\mathcal{A}_2\in B_*}\|\xi\otimes(U_{\mathcal{A}_1}-U_{\mathcal{A}_2})\|_2^2\le\frac{32\tilde{\eta}^2(d+1)m}{\kappa^2}\left(\inf_{\mathcal{A}\in B_*}\|\Sigma_{\mathcal{A}}\|_{op}\right)\|\mathcal{A}_1-\mathcal{A}_2\|_F^2\right)\le c\delta.$$

Proof. For each $j,\ell\in\{1,\dots,m/k\}$, define the event

$$E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2}:=\{\lambda_1(\mathcal{A}_1^{(j)}[\xi])=\max\ \text{and}\ \lambda_1(\mathcal{A}_2^{(\ell)}[\xi])=\max\}.$$

Now assume that $E_{j,j}$ holds for some j. Observe that $\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F^2=2\sin(\Theta(u_1,u_2))^2$, where $u_1,u_2\in S^{k-1}$ are the leading eigenvectors of $\mathcal{A}_1^{(j)}[\xi]$ and $\mathcal{A}_2^{(j)}[\xi]$, respectively. Also, note that for all $\mathcal{A}\in B(\mathcal{A}_*,\kappa/4)$, Weyl's inequality implies

$$\lambda_1(\mathcal{A}^{(j)}[u])-\lambda_2(\mathcal{A}^{(j)}[u])\ge\lambda_1(\mathcal{A}_*^{(j)}[u])-\lambda_2(\mathcal{A}_*^{(j)}[u])-2\|\mathcal{A}^{(j)}-\mathcal{A}_*^{(j)}\|_{op}\ge\frac{\kappa}{2}$$

for all $u\in S^d$. By a variation of the Davis–Kahan theorem (Corollary 1 in [33]) and (3.6),

$$\sin(\Theta(u_1,u_2))\le\frac{2\|(\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)})[\xi]\|_{op}}{\lambda_1(\mathcal{A}_2^{(j)}[\xi])-\lambda_2(\mathcal{A}_2^{(j)}[\xi])}\le\frac{4\|\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)}\|_F}{\kappa}.$$

On the events $E_{j,\ell}$ where $\ell\neq j$, we have the upper bound $\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F\le 2$. Together this implies the following general upper bound:

$$\begin{aligned}\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F&=\sum_{j=1}^{m/k}\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F 1_{E_{j,j}^{\mathcal{A}_1,\mathcal{A}_2}}+\sum_{j=1}^{m/k}\sum_{\ell\neq j}\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F 1_{E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2}}\\ &\le\frac{2^{5/2}}{\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F 1_{E_{j,j}^{\mathcal{A}_1,\mathcal{A}_2}}+2\sum_{j=1}^{m/k}\sum_{\ell\neq j}1_{E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2}}.\end{aligned}\tag{A.12}$$

We now bound the probability of $E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2}$. By Weyl's inequality,

$$\begin{aligned}E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2}&\subseteq\left\{\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])\ge\lambda_1(\mathcal{A}_2^{(j)}[\xi])\ \text{and}\ \lambda_1(\mathcal{A}_1^{(j)}[\xi])\ge\lambda_1(\mathcal{A}_1^{(\ell)}[\xi])\right\}\\ &\subseteq\left\{\left(\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right)\left(\lambda_1(\mathcal{A}_1^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_1^{(j)}[\xi])\right)\le 0\right\}.\end{aligned}$$

Then, by the fact that $(a-b)^2\ge a^2$ if $ab<0$ and again by Weyl's inequality,

(A.13)
$$\begin{aligned}&E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2}\\ &\subseteq\left\{\left(\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_1^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])+\lambda_1(\mathcal{A}_1^{(j)}[\xi])\right)^2\ge\left(\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right)^2\right\}\\ &\subseteq\left\{\left|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_1^{(\ell)}[\xi])\right|+\left|\lambda_1(\mathcal{A}_1^{(j)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right|\ge\left|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right|\right\}\\ &\subseteq\left\{\|\mathcal{A}_2^{(\ell)}[\xi]-\mathcal{A}_1^{(\ell)}[\xi]\|_{op}+\|\mathcal{A}_1^{(j)}[\xi]-\mathcal{A}_2^{(j)}[\xi]\|_{op}\ge\left|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right|\right\}\\ &\subseteq\left\{\tilde{\eta}\sqrt{d+1}\left(\|\mathcal{A}_2^{(\ell)}-\mathcal{A}_1^{(\ell)}\|_F+\|\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)}\|_F\right)\ge\left|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right|\right\}.\end{aligned}$$

Thus, by assumption (3.4),

$$\begin{aligned}&\mathbb{P}(E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2})\\ &\leq \mathbb{P}\left(\tilde{\eta}\sqrt{d+1}\left(\|\mathcal{A}_2^{(\ell)}-\mathcal{A}_1^{(\ell)}\|_F+\|\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)}\|_F\right)\geq \left|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right|\right)\\ &\leq c\tilde{\eta}\sqrt{d+1}\frac{\left(\|\mathcal{A}_2^{(\ell)}-\mathcal{A}_1^{(\ell)}\|_F+\|\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)}\|_F\right)}{\mathbb{E}\left[\left|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right|\right]}. \end{aligned} \tag{A.14}$$

By Weyl's inequality and the triangle inequality,

$$|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])|\geq |\lambda_1(\mathcal{A}_*^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_*^{(j)}[\xi])|-\frac{k\kappa}{2m}, \tag{A.15}$$

and by assumption (3.5),

$$\mathbb{E}\left[|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])|\right]\geq \frac{m\kappa}{k\delta}-\frac{k\kappa}{2m}\geq\frac{\kappa}{2}. \tag{A.16}$$

In order to prove (A.10), we see that the bounds (A.12), (A.14), and (A.16) imply

$$\begin{aligned}&\mathbb{E}[\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F]\\ &\leq \frac{2^{5/2}}{\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F+2\sum_{j=1}^{m/k}\sum_{\ell\neq j}\mathbb{P}(E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2})\\ &\leq \frac{2^{5/2}}{\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F+\frac{4c\tilde{\eta}\sqrt{d+1}}{\kappa}\sum_{j=1}^{m/k}\sum_{\ell\neq j}\left(\|\mathcal{A}_2^{(\ell)}-\mathcal{A}_1^{(\ell)}\|_F+\|\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)}\|_F\right)\\ &= \frac{2^{5/2}}{\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F+\frac{8mc\tilde{\eta}\sqrt{d+1}}{k\kappa}\sum_{j=1}^{m/k}\|\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)}\|_F\\ &\leq \frac{2^{5/2}}{\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F+\frac{8m^{3/2}c\tilde{\eta}}{k^{3/2}\kappa}\left(\sum_{j=1}^{m/k}\|\mathcal{A}_1^{(j)}-\mathcal{A}_2^{(j)}\|_F^2\right)^{1/2}\\ &\leq \frac{16m^{3/2}\max\{1,c\}\tilde{\eta}\sqrt{d+1}}{k^{3/2}\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F,\end{aligned}$$

where we have used the inequality $\|x\|_1\leq\sqrt{n}\|x\|_2$ for $x\in\mathbb{R}^n$. Next we prove claim (A.11). First, we see that by (A.13)

$$\begin{aligned}&\mathbb{P}\left(\cup_{\mathcal{A}_1,\mathcal{A}_2\in B_*}E_{j,\ell}^{\mathcal{A}_1,\mathcal{A}_2}\right)\\ &\leq \mathbb{P}\left[\cup_{\mathcal{A}_1,\mathcal{A}_2\in B_*}\left\{\tilde{\eta}\sqrt{d+1}\left(\|\mathcal{A}_2^{(\ell)}-\mathcal{A}_1^{(\ell)}\|_F+\|\mathcal{A}^{(j)}-\mathcal{A}_2^{(j)}\|_F\right)\geq\left|\lambda_1(\mathcal{A}_2^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_2^{(j)}[\xi])\right|\right\}\right],\end{aligned}$$

and thus by the union bound, (A.15), and assumption (3.5),

$$\begin{aligned}&\mathbb{P}\left(\cup_{\mathcal{A}_1,\mathcal{A}_2\in B_*}\cup_{j\neq\ell}E_{i,j}^{\mathcal{A}_1,\mathcal{A}_2}\right)\leq\sum_{j\neq\ell}\mathbb{P}\left(\cup_{\mathcal{A}_1,\mathcal{A}_2\in B_*}E_{i,j}^{\mathcal{A}_1,\mathcal{A}_2}\right)\\ &\quad\leq\frac{m^2}{k^2}\mathbb{P}\left[\frac{k\kappa}{m}\geq\left|\lambda_1(\mathcal{A}_*^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_*^{(j)}[\xi])\right|\right]\\ &\quad\leq\frac{m^2}{k^2}\mathbb{P}\left[\frac{k^2\delta}{m^2}\mathbb{E}\left[\left|\lambda_1(\mathcal{A}_*^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_*^{(j)}[\xi])\right|\right]\geq\left|\lambda_1(\mathcal{A}_*^{(\ell)}[\xi])-\lambda_1(\mathcal{A}_*^{(j)}[\xi])\right|\right]\leq c\delta.\end{aligned}$$

Thus, (A.12) implies

$$\mathbb{P}\left(\bigcup_{\mathcal{A}_1,\mathcal{A}_2\in B_*}\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F \geq \frac{2^{5/2}}{\kappa}\|\mathcal{A}_1-\mathcal{A}_2\|_F\right)\leq c\delta.$$

The inequality $\|\xi\otimes(U_{\mathcal{A}_1}-U_{\mathcal{A}_2})\|_2\leq\tilde{\eta}\sqrt{d+1}\|U_{\mathcal{A}_1}-U_{\mathcal{A}_2}\|_F$ and Lemma A.1 give the second claim. ∎

LEMMA A.6. *Define* $B_q(z,R):=\{x\in\mathbb{R}^q:\|x-z\|_2\leq R\}$ *for* $R>0$ *and* $z\in\mathbb{R}^q$. *Let* $\{\xi_a\}_{a\in B_q(z,R)}$ *be a stochastic process in* $\mathbb{R}^d$ *such that*

(i) $\|\xi_a\|_2\leq\eta\|\Sigma_a^{1/2}\|_{op}$;

(ii) *for all* $a_1,\ a_2\in B_q(z,R)$,

$$\mathbb{P}\left[\bigcup_{a_1,a_2\in B_q(z,R)}\|\xi_{a_1}-\xi_{a_2}\|_2\geq K\left(\inf_{a\in B_q(z,R)}\|\Sigma_a\|_{op}^{1/2}\right)\|a_1-a_2\|_2\right]\leq\delta,$$

where $\Sigma_a=\mathbb{E}[\xi_a\xi_a^T]$ *for all* $a\in B_q(z,R)$. *Define* Ξ_a *as in Lemma* A.3 *and fix* $\tau\in(0,1)$. *Then, there exist constants* c_0,c_1 *such that for* $n\geq c_0\eta^3RK/\tau^4$, *with probability greater than* $1-2de^{-c_1n\tau^2/\eta^2}-\delta$,

$$1-\tau\leq\inf_{a\in B_q(z,R)}\frac{\lambda_{\min}\left(\Xi_a^T\Xi_a\right)}{n\lambda_{\max}(\Sigma_a)}\leq\sup_{a\in B_q(z,R)}\frac{\lambda_{\max}\left(\Xi_a^T\Xi_a\right)}{n\lambda_{\max}(\Sigma_a)}\leq 1+\tau.$$

Proof. As in Lemma A.3, first suppose that for all a, ξ_a is isotropic. Fix $\tau>0$. For each a, the Matrix Bernstein's inequality [29, Theorem 1.6.2] implies, for all $s\geq 0$,

$$\mathbb{P}\left(\left\|\frac{1}{n}\Xi_a^T\Xi_a-I\right\|_{op}\geq s\right)\leq 2d\exp\left(\frac{-ns^2}{2\eta^2(1+s)}\right).$$

Now observe, by the reverse triangle inequality, that for any $a,b\in B_q(z,R)$,

$$\begin{aligned}
&\left|\left\|\frac{1}{n}\Xi_a^T\Xi_a-I\right\|_{op}-\left\|\frac{1}{n}\Xi_b^T\Xi_b-I\right\|_{op}\right|^2\\
&\leq\frac{1}{n}\|\Xi_a^T\Xi_a-\Xi_b^T\Xi_b\|_{op}^2\\
&\leq\frac{1}{n}\sup_{x\in\mathbb{S}^{d-1}}\sum_{i=1}^n\langle\xi_a^{(i)}-\xi_b^{(i)},x\rangle\langle\xi_a^{(i)}+\xi_b^{(i)},x\rangle\\
&\leq\frac{1}{n}\sum_{i=1}^n\|\xi_a^{(i)}-\xi_b^{(i)}\|_2\|\xi_a^{(i)}+\xi_b^{(i)}\|_2\leq\frac{2\eta}{n}\sum_{i=1}^n\|\xi_a^{(i)}-\xi_b^{(i)}\|_2.
\end{aligned}$$

Now, let $\mathcal{M}_\tau$ be a $\frac{\tau^2}{16\eta K}$-net in the ball $B_q(z,R)$. For $a\in B_q(z,R)$, let $a_\tau\in\mathcal{M}_\tau$ be the parameter such that $\|a-a_\tau\|_2\leq\frac{\tau^2}{16\eta K}$. Then,

$$\begin{aligned}
&\sup_{a\in B_q(z,R)}\left\|\frac{1}{n}\Xi_a^T\Xi_a-I\right\|_{op}\\
&\quad\leq\sup_{a\in B_q(z,R)}\left|\left\|\frac{1}{n}\Xi_a^T\Xi_a-I\right\|_{op}-\left\|\frac{1}{n}\Xi_{a_\tau}^T\Xi_{a_\tau}-I\right\|_{op}\right|+\sup_{a\in\mathcal{M}_\tau}\left\|\frac{1}{n}\Xi_a^T\Xi_a-I\right\|_{op}
\end{aligned}$$

$$\leq \sup_{a\in B_q(z,R)} \left(\frac{2\eta}{n}\sum_{i=1}^n \|\xi_a^{(i)} - \xi_{a_\tau}^{(i)}\|_2\right)^{1/2} + \sup_{a\in\mathcal{M}_\tau} \left\|\frac{1}{n}\Xi_a^T\Xi_a - I\right\|_{op}.$$

By (A.1), $|\mathcal{M}_\tau| \leq (1+\frac{32\eta RK}{\tau^2})^q$. A union bound then gives

$$\begin{aligned}
&\mathbb{P}\left(\sup_{a\in B_q(z,R)} \left\|\frac{1}{n}\Xi_a^T\Xi_a - I\right\|_{op} \geq \tau\right) \\
&\quad\leq \mathbb{P}\left(\sup_{a\in\mathcal{M}_{\tau,K}} \left\|\frac{1}{n}\Xi_a^T\Xi_a - I\right\|_{op} \geq \frac{\tau}{2}\right) + \mathbb{P}\left(\sup_{a\in B_q(z,R)} \frac{2\eta}{n}\sum_{i=1}^n \|\xi_a^{(i)} - \xi_{a_\tau}^{(i)}\|_2 \geq \frac{\tau^2}{4}\right) \\
&\quad\leq 2d\left(1+\frac{32\eta RK}{\tau^2}\right)^q \exp\left(\frac{-n\tau^2}{8\eta^2(1+\tau/2)}\right) \\
&\qquad + \mathbb{P}\left(\sup_{a\in B_q(z,R)} \frac{1}{n}\sum_{i=1}^n \|\xi_a^{(i)} - \xi_{a_\tau}^{(i)}\|_2 \geq \frac{\tau^2}{8\eta}\right).
\end{aligned}$$

To bound the second probability, we see that by the assumptions on ξ_a,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{a\in B_q(z,R)} \frac{1}{n}\sum_{i=1}^n \|\xi_a^{(i)} - \xi_{a_\tau}^{(i)}\|_2 \geq \frac{\tau^2}{8\eta}\right) \\
&\leq \mathbb{P}\left(\sup_{a\in B_q(z,R)} \frac{1}{n}\sum_{i=1}^n \|\xi_a^{(i)} - \xi_{a_\tau}^{(i)}\|_2 \geq \frac{\tau^2}{8\eta}, \bigcap_{a\in B_q(z,R)} \|\xi_a^{(i)} - \xi_{a_\tau}^{(i)}\|_2 \leq K\|a-a_\tau\|_2\right) \\
&\qquad + \mathbb{P}\left(\bigcup_{a\in B_q(z,R)} \|\xi_a^{(i)} - \xi_{a_\tau}^{(i)}\|_2 \geq K\|a-a_\tau\|_2\right) \leq \delta.
\end{aligned}$$

Thus, for $n \geq c_2\eta^3 RK/\tau^4$,

$$\mathbb{P}\left(\sup_{a\in B_q(z,R)} \left\|\frac{1}{n}\Xi_a^T\Xi_a - I\right\|_{op} \geq \tau\right) \leq 2de^{-\frac{c_3 n\tau^2}{\eta^2}} + \delta. \qquad \square$$

LEMMA A.7. *Consider the setting of Theorem* 3.2 *and let* $B_* := B(\mathcal{A}_*, \frac{k\kappa}{4m\tilde{\eta}\sqrt{d+1}})$. *Define* $\mathcal{P}$ *as in Lemma* A.4. *Then, there exist absolute constants* c_i, $i=0,\dots,3$, *such that for* $n \geq c_0\tilde{\eta}^2(d+1)^{3/2}mk$,

$$\mathbb{P}\left(\sup_{P\in\mathcal{P}} \|P\varepsilon\|_2^2 \geq c_1\sigma^2\log(n)^2(d+1)m^2\right) \leq n^{-\frac{c_4 m}{\tilde{\eta}^2}\min\left\{\frac{(d+1)m}{\sigma^2},1\right\}} + 2de^{-\frac{c_3 n}{\tilde{\eta}^2(d+1)m}} + c\delta.$$

Proof. We proceed as in the proof of Lemma A.4 to show that for $t>0$,

$$\text{(A.17)}\quad \mathbb{P}(\sup_{P\in\mathcal{P}} \|P\varepsilon\|^2 \geq t) \leq \mathbb{P}\left(\sup_{\mathcal{A}\in B_*} \|\Xi_{\mathcal{A}}^T\varepsilon\|_2^2 \geq \frac{tn}{2m}\right) + \mathbb{P}\left(\inf_{\mathcal{A}\in B_*} \frac{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}{n\|\Sigma_{\mathcal{A}}\|_{op}} \leq \frac{1}{2}\right).$$

To upper bound the second probability above, Lemmas A.5 and A.6 applied to the stochastic process $\xi_a := \xi \otimes U_{\mathcal{A}}$ imply that for $n \geq c_1\tilde{\eta}^3(d+1)^{3/2}mk$,

$$\text{(A.18)}\qquad \mathbb{P}\left(\inf_{\mathcal{A}\in B_*} \frac{\lambda_{\min}(\Xi_{\mathcal{A}}^T\Xi_{\mathcal{A}})}{n\|\Sigma_{\mathcal{A}}\|_{op}} \leq \frac{1}{2}\right) \leq 2de^{-\frac{c_2 n}{\tilde{\eta}^2(d+1)m}} + c\delta.$$

For the first probability, proceeding as in the Lemma A.4, we obtain the following: For $s \geq \max\{c_0\sigma\sqrt{\log(n)}\mathbb{E}[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|_2], c_0^2\sigma^2\log(n)n(d+1)\}$,

(A.19)

$$\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2 - n(d+1)\sigma^2 \geq s\right) \leq e^{-\frac{c_0}{\sigma^2\log(n)}\min\left\{\frac{s^2}{\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|\right]^2}, \frac{s}{\tilde{\eta}^2 n(d+1)}\right\}}.$$

Now, to upper bound $\mathbb{E}[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|_2]$, we first observe that by the independence of the covariates and noise,

$$\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|\right] \leq \mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2\right]^{1/2} = \mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}\|_F^2\|\varepsilon\|^2\right]^{1/2}$$
$$= \sigma\sqrt{n}\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}\|_F^2\right]^{1/2}.$$

For all $\mathcal{A}\in B_*$,

$$\|\Xi_{\mathcal{A}}\|_F^2 = \sum_{i=1}^n \|\xi^{(i)}\otimes U_{\mathcal{A}}^{(i)}\|_2^2 \leq \tilde{\eta}^2(d+1)n,$$

and thus

$$\mathbb{E}\left[\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|\right] \leq n\tilde{\eta}\sigma\sqrt{d+1}.$$

Then, for $s \geq \max\{c_0\sigma^2\tilde{\eta}n\sqrt{\log(n)(d+1)}, c_0^2\sigma^2 n\log(n)(d+1)\}$

$$\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2 - n(d+1)\sigma^2 \geq s\right)$$
$$\leq \exp\left(-\frac{c_1}{\sigma^2\log(n)}\min\left\{\frac{s^2}{\tilde{\eta}^2\sigma^2 n^2(d+1)}, \frac{s}{\tilde{\eta}^2 n(d+1)}\right\}\right).$$

Letting $t = c_9\sigma^2\log(n)^2(d+1)m^2$ for a constant $c_9 > 0$ large enough,

$$\mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2 \geq \frac{tn}{2m}\right) = \mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2 \geq c_2\sigma^2 n\log(n)^2(d+1)m\right)$$
$$\leq \mathbb{P}\left(\sup_{\mathcal{A}\in B_*}\|\Xi_{\mathcal{A}}^T\varepsilon\|^2 - n(d+1)\sigma^2 \geq c_3\sigma^2 n\log(n)^2(d+1)m\right)$$
$$\leq \exp\left(-c_4\log(n)\min\left\{\frac{(d+1)m^2}{\tilde{\eta}^2\sigma^2}, \frac{m}{\tilde{\eta}^2}\right\}\right) \leq n^{-\frac{c_4 m}{\tilde{\eta}^2}\min\left\{\frac{(d+1)m}{\sigma^2}, 1\right\}}.$$

Finally, combining the above bound with (A.17) gives, for $n \geq c_1\tilde{\eta}^3(d+1)^{3/2}mk$,

$$\mathbb{P}\left(\sup_{P\in\mathcal{P}}\|P\varepsilon\|_2^2 \geq c_3\sigma^2\log(n)^2(d+1)m^2\right)$$
$$\leq n^{-\frac{c_4 m}{\tilde{\eta}^2}\min\left\{\frac{(d+1)m}{\sigma^2}, 1\right\}} + 2de^{-\frac{c_3 n}{\tilde{\eta}^2(d+1)m}} + c\delta. \qquad \square$$

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