Free Descriptions of Convex Sets

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Abstract

Convex sets arising in a variety of applications are well-defined for every relevant dimension. Examples include the simplex and the spectraplex that correspond to probability distributions and to quantum states; combinatorial polytopes and their relaxations such as the cut polytope and the elliptope in integer programming; and unit balls of regularizers such as the ℓ_p and Schatten norms in inverse problems. Moreover, these sets are often specified using conic descriptions that can be obviously instantiated in any dimension. We develop a systematic framework to study such free descriptions of convex sets. We show that free descriptions arise from a recently-identified phenomenon in algebraic topology called representation stability, which relates invariants across dimensions in a sequence of group representations. Our framework yields structural results for free descriptions pertaining to the relations between the sets they describe across dimensions, extendability of a single set in a given dimension to a freely-described sequence, and continuous limits of such sequences. We also develop a procedure to obtain parametric families of freely-described convex sets whose structure is adapted to a given application; illustrations are provided via examples that arise in the literature as well as new families that are derived using our procedure. We demonstrate the utility of our framework in two contexts. First, we develop an algorithm for a free analog of the convex regression problem, where a convex function is fit to input-output data; by searching over our parametric families, we can fit a function to low-dimensional inputs and extend it to any other dimension. Second, we prove that many sequences of symmetric conic programs can be solved in constant time, which unifies and strengthens several results in the literature.

Keywords: cone programming, convex optimization, free spectrahedra, graphons, representation theory

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1 Introduction

Convex sets play a central role in numerous areas of the mathematical sciences such as optimization, statistical inference, control, inverse problems, and information theory. The convex sets arising in all these domains are often well-defined in every relevant dimension. Indeed, unit balls of standard regularizers used in inverse problems (e.g., ℓ_p or Schatten norms) are defined for vectors and matrices of any size. Polytopes associated to graph problems and their relaxations (e.g., the cut polytope and the elliptope approximating it) are defined for graphs of any size. Information-theoretic quantities (e.g., relative entropy) as well as their quantum analogues, are defined for distributions and states on any number of (qu)bits. Consequently, all these convex sets and many others should be viewed not as single sets but as sequences indexed by dimension. Furthermore, there is often a single "free" description of all the sets in such a sequence, as the following example illustrates.

Example 1.1. *Here are a few examples of sequences of convex sets and their descriptions.*

(a) The simplex in n dimensions is $\Delta^{n-1} = \{x \in \mathbb{R}^n : x \geq 0,\ \mathbb{1}_n^\top x = 1\}$, which is the set of probability distributions over n items. Here $x \geq 0$ denotes an entrywise nonnegative vector x, and $\mathbb{1}_n \in \mathbb{R}^n$ is the vector of all-1's.

(b) The spectraplex, or the set of density matrices, of size n is $\mathcal{D}^{n-1} = \{X \in \mathbb{S}^n : X \succeq 0,\ \mathrm{Tr}(X) = 1\}$. It is the set of density matrices describing mixed states in quantum mechanics [1, §2.4]. Here $X \succeq 0$ denotes a symmetric positive-semidefinite (PSD) matrix X.

(c) The ℓ_2 ball in $\mathbb{R}^n$ is given by $\mathcal{B}_{\ell_2}^n = \left\{x \in \mathbb{R}^n : \begin{bmatrix} 1 & x^\top \\ x & I_n \end{bmatrix} \succeq 0\right\}$ where I_n denotes the $n \times n$ identity.

(d) The elliptope of size n is $\{X \in \mathbb{S}^n : X \succeq 0, \mathrm{diag}(X) = \mathbb{1}_n\}$. It arises in a standard relaxation of the max-cut problem [2].

Note that all of the above descriptions are clearly defined for any relevant dimension. Indeed, these descriptions are composed of elements such as $\mathbb{1}_n$ and I_n, linear maps such as $\mathrm{diag}(\cdot)$ and $\mathrm{Tr}(\cdot)$, and inequalities such as $\geq$ and $\succeq$, all of which are *"free"* in the sense that they are well-defined in every relevant dimension. In this paper, we develop a systematic framework to study such *freely-described convex sets*.

One motivation for this effort is to facilitate structural understanding of convex sets that can be instantiated in any dimension and of sequences of optimization problems over such sets. Such sequences arise in several applications. For example, in extremal combinatorics [3, 4] it is of interest to certify inequalities between graph homomorphism densities involving graphs of every size. In quantum information and control theory [5, 6], many problems reduce to optimizing (traces of) polynomials in matrix variables of every size. Despite the ubiquity of sequences of convex sets in these problem domains, the existence and interplay between the sets in different dimensions is typically not explicitly discussed or exploited in the literature. In the present paper, we explicitly consider sequences of convex sets that are related in a concrete fashion across dimensions, and we present a framework to derive several structural results about such sets as well as optimization programs over them; as one notable illustration, we show that sequences of invariant conic programs (including some arising in the above applications) can be solved in time independent of dimension.

A second motivation for our effort stems from the growing interest in learning solution methods for various problem families from data. Here one is typically given input-output data, and the goal is to fit a mapping that approximately fits the data. This framework has been fruitfully applied to domains including integer programming, inverse problems, and numerical solvers for PDEs [7, 8, 9, 10, 11, 12]. However, a fundamental limitation in much of this literature is that the mappings learned from data are typically only defined for inputs of the same dimension as the ones in the training data, and extension to inputs of different dimensions is handled on a case-by-case basis. In contrast, we wish to learn *algorithms* that should be defined for inputs of any relevant size, i.e., we aim to identify a sequence of solution maps, one for each input size. Identifying a freely-described convex set offers a convenient approach to learning an algorithm specified as linear optimization over convex sets (convex programs). Further, to facilitate numerical search over a family of freely-described convex sets—for example, to fit an element of this family to training data—it is natural to seek *finitely-parametrized* families of such sets, such as the following examples.

Example 1.2. *The following are examples of finitely-parametrized families of freely-described convex sets.*

(a) The sequence of subsets $\{\alpha_1 \mathrm{diag}(X) + \alpha_2 X\mathbb{1}_n : X \succeq 0,\ \alpha_3 \mathrm{Tr}(X)\mathbb{1}_n + \alpha_4 \mathrm{diag}(X) + \alpha_5 X\mathbb{1}_n = \alpha_6 \mathbb{1}_n\}$ of $\mathbb{R}^n$ is parametrized by $\alpha \in \mathbb{R}^6$.

(b) Free spectrahedra are sequences of the form $\left\{(X_1, \ldots, X_d) \in (\mathbb{S}^n)^d : L_0 \otimes I_n + \sum_{i=1}^d L_i \otimes X_i \succeq 0\right\}$ that are parametrized by $L_0, \ldots, L_d \in \mathbb{S}^k$ for $k \in \mathbb{N}$. They arise in the theory of matrix convexity and free convex algebraic geometry [13, 14].

(c) The sequence $\mathcal{C}_n = \Big\{X \in \mathbb{S}^n : \frac{\mathbb{1}_n^\top X \mathbb{1}_n}{n^2} L_1 \otimes \mathbb{1}_n\mathbb{1}_n^\top + \frac{\mathrm{Tr}(X)}{n} L_2 \otimes \mathbb{1}_n\mathbb{1}_n^\top + L_3 \otimes \frac{1}{n}\left(X\mathbb{1}_n\mathbb{1}_n^\top + \mathbb{1}_n\mathbb{1}_n^\top X\right) +$

$L_4 \otimes \left(\mathrm{diag}(X)\mathbb{1}_n^\top + \mathbb{1}_n \mathrm{diag}(X)^\top\right) + L_5 \otimes X + L_6 \otimes \mathbb{1}_n\mathbb{1}_n^\top + L_7 \otimes (nI_n) \succeq 0\Big\}$ is parametrized by

$L_1, \ldots, L_7 \in \mathbb{S}^k$. *We obtain this sequence from our framework in Section 4.7 by viewing a matrix* $X \in \mathbb{S}^n$ *representing a weighted graph as a step* graphon *[15], and deriving a parametric family of free descriptions extending continuously to more general spaces of graphons.*

Each example here is a parametric family, and constitutes a freely-described sequence of convex sets for each value of its associated parameters.

Note that the parametric free descriptions in Example 1.2 are composed of linear combinations of the free elements and linear maps that appear in the descriptions of Example 1.1. In fact, the parameters above are simply the coefficients in these linear combinations. Thus, we can obtain finitely-parametrized families of freely-described convex sets by deriving *finite-dimensional spaces* of free elements.

The central thesis of this paper is that the free elements which constitute free descriptions arise from a recently-identified phenomenon in algebraic topology called *representation stability*, which relates invariants across dimensions in a sequence of group representations [16]. We use these relations to formally define the free elements that constitute our free descriptions, and investigate various structural properties of such descriptions as well as the associated sequences of optimization problems. Our definitions together with results from representation stability also yield finite-dimensional spaces of free elements, which we use to derive parametric families of freely-described convex sets adapted to specific applications and to fit elements of these families to data. We outline the contributions of this paper in the remainder of the introduction. We do not assume prior familiarity with representation stability, and the relevant background is presented in Section 2.

1.1 Our Framework and Contributions

This paper consists of four main contributions. First, we formally define free descriptions of convex sets by generalizing the insights derived from Examples 1.1 and 1.2 using representation stability. Second, we prove structural results pertaining to freely-described sequences of convex sets, the relationship between sets in such a sequence in different dimensions, and limits of such sequences. Our proofs combine concepts from convex analysis and representation theory. Third, we derive consequences of our framework for invariant conic programs, showing that sequences of such programs of growing dimension can often be solved in constant time. Our results unify and generalize existing work in the literature. Finally, we apply our framework and its structural results to derive an algorithm for fitting a freely-described convex function to given data in different dimensions, a problem we call free convex regression. The latter two contributions are aimed at addressing our mathematical and algorithmic motivations above.

1.1.1 Freely-described Convex Sets

To formalize free descriptions of convex sets, we begin by briefly reviewing *conic descriptions*, which express a convex set as an affine section of a convex cone. Conic descriptions are the most popular approach to specifying convex sets in the optimization literature, and they have played a central role in the development of modern convex optimization [17]. Indeed, we often classify convex sets based on their conic descriptions—polyhedra are affine sections of nonnegative orthants and spectrahedra are affine sections of PSD cones. Formally, if $\mathbb{V}, \mathbb{W}, \mathbb{U}$ are (finite-dimensional) vector spaces and $\mathcal{K} \subseteq \mathbb{U}$ is a convex cone, then a convex subset $\mathcal{C} \subseteq \mathbb{V}$ can be described using linear maps $A\colon \mathbb{V} \to \mathbb{U}$, $B\colon \mathbb{W} \to \mathbb{U}$ and a vector $u \in \mathbb{U}$ as follows:

$$\mathcal{C} = \{x \in \mathbb{V} : \exists y \in \mathbb{W} \text{ s.t. } Ax + By + u \in \mathcal{K}\}. \tag{Conic}$$

We will refer to the spaces $\mathbb{W}$ and $\mathbb{U}$ as the *description spaces* associated to the conic description. If the cone $\mathcal{K}$ is a nonnegative orthant (resp., PSD cone), then linear optimization over $\mathcal{C}$ is a linear (resp., semidefinite) program. The type of the cone as well as its dimension determine the computational complexity of optimization over $\mathcal{C}$.

Each sequence of convex sets $\mathcal{C}_n \subseteq \mathbb{V}_n$ in Examples 1.1 and 1.2 is given by (Conic) for suitable sequences of description spaces $\mathbb{W}_n, \mathbb{U}_n$, of vectors $u_n \in \mathbb{U}_n$, of linear maps $A_n\colon \mathbb{V}_n \to \mathbb{U}_n$, $B_n\colon \mathbb{W}_n \to \mathbb{U}_n$, and of convex cones $\mathcal{K}_n \subseteq \mathbb{U}_n$. In particular, all the sequences of cones $\mathcal{K}_n$ in those examples are standard sequences such as the nonnegative orthants and PSD cones. Expressing convex sets in terms of such standard sequences has the benefit that optimization over these sets can be performed using standard off-the-shelf software. Thus, all the sequences of cones, as well as the vector spaces containing them, appearing in this paper come from such standard sequences.

Having fixed standard sequences of vector spaces and cones, we seek to formalize the free vectors u_n and linear maps A_n, B_n that are defined for any dimension, such as those appearing in Examples 1.1 and 1.2. Furthermore, we wish to obtain finite-dimensional spaces of these free elements, which yield finitely-parametrized families of free descriptions. To that end, we observe that these free elements are sequences of invariants under sequences of groups related in a particular way across dimensions. For example, the all-1's vector $\mathbb{1}_n$ of length n is invariant under the group of permutations on n letters acting by permuting coordinates. Further, the all-1's vectors of different lengths are related to each other: extracting the first n entries of $\mathbb{1}_{n+1}$ yields $\mathbb{1}_n$. Similarly, the $n \times n$ identity matrix I_n is invariant under the orthogonal group of size n acting by conjugation, and extracting the top left $n \times n$ submatrix of I_{n+1} yields I_n. Thus, to give a formal definition for free vectors and linear maps, we consider sequences of groups acting on sequences of vector spaces, and we require the spaces in the sequence to be related to each other – specifically, we embed lower-dimensional spaces into higher-dimensional ones and project higher-dimensional spaces onto lower-dimensional ones. Such sequences of group representations are called *consistent sequences* and they were first defined in the seminal paper [16] introducing representation stability.

Definition 1.3 (Consistent sequences [16])**.** *Fix a family of compact*[1] *groups* $\mathscr{G} = \{\mathsf{G}_n\}_{n\in\mathbb{N}}$ *such that* $\mathsf{G}_n \subseteq \mathsf{G}_{n+1}$*. A* consistent sequence *of* $\mathscr{G}$*-representations is a sequence* $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}_{n\in\mathbb{N}}$ *satisfying the following properties:*

(a) $\mathbb{V}_n$ *is an orthogonal* G_n*-representation;*

(b) $\varphi_n \colon \mathbb{V}_n \hookrightarrow \mathbb{V}_{n+1}$ *is a linear* G_n*-equivariant isometry.*

Unless we want to emphasize the embeddings φ_n*, we shall identify* $\mathbb{V}_n$ *with its image inside* $\mathbb{V}_{n+1}$*. We then write* $\mathscr{V} = \{\mathbb{V}_n\}$ *and take* φ_n *to be inclusions* $\mathbb{V}_n \subseteq \mathbb{V}_{n+1}$*.*[2]

As φ_n is an isometry, its adjoint φ_n^* defines the orthogonal projection $\mathcal{P}_{\mathbb{V}_n}$ of higher-dimensional spaces onto lower-dimensional ones. We now formally define free vectors or linear maps as sequences of invariants projecting onto each other.

Definition 1.4 (Freely-described elements)**.** *A* freely-described element *in a consistent sequence* $\{(\mathbb{V}_n, \varphi_n)\}$ *of* $\{\mathsf{G}_n\}$*-representations is a sequence* $\{v_n \in \mathbb{V}_n^{\mathsf{G}_n}\}$ *of invariants satisfying* $\varphi_n^*(v_{n+1}) = v_n$ *for all* n*.*

Importantly, the set of freely-described elements in a given consistent sequence $\mathscr{V}$ is naturally a linear space,[3] because if $\{v_n\}$ and $\{v_n'\}$ are freely-described elements then so is $\{\alpha v_n + \beta v_n'\}$ for any $\alpha, \beta \in \mathbb{R}$. Furthermore, fundamental results in representation stability imply that these spaces of freely-described elements are often finite-dimensional. More precisely, when the consistent sequence $\mathscr{V}$ is *finitely-generated*, the restricted projections $\mathcal{P}_{\mathbb{V}_n}|_{\mathbb{V}_{n+1}^{\mathsf{G}_{n+1}}} \colon \mathbb{V}_{n+1}^{\mathsf{G}_{n+1}} \to \mathbb{V}_n^{\mathsf{G}_n}$ become isomorphisms for all n exceeding the *presentation degree* of the sequence, see Section 2 for precise definitions.

Definition 1.4 also defines freely-described linear maps. Indeed, if $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$ and $\mathscr{U} = \{(\mathbb{U}_n, \psi_n)\}$ are consistent sequences of $\{\mathsf{G}_n\}$-representations, then the sequence of spaces of linear maps between them can be naturally identified with $\{(\mathbb{V}_n \otimes \mathbb{U}_n, \varphi_n \otimes \psi_n)\}$ where $\otimes$ is the tensor (or Kronecker) product, and this is also a consistent sequence. A freely-described element of $\mathscr{V} \otimes \mathscr{U}$ is a sequence of equivariant maps $\{A_n \in \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n}\}$ satisfying $\psi_n^* A_{n+1} \varphi_n = A_n$. When φ_n, ψ_n are inclusions, this simplifies to $\mathcal{P}_{\mathbb{U}_n} A_{n+1}|_{\mathbb{V}_n} = A_n$ where $\mathcal{P}_{\mathbb{U}_n}$ is orthogonal projection onto $\mathbb{U}_n$. From results in representation stability, it follows that for a large class of consistent sequences, the sequence $\mathscr{V} \otimes \mathscr{U}$ is finitely-generated when $\mathscr{V}, \mathscr{U}$ are, and the presentation degree of the former can be bounded in terms of those of the latter; see Theorem 2.11. Thus, the space of freely-described linear maps is often finite-dimensional as well.

Example 1.5 (Vectors with zero-padding)**.** *Let* $V_n = \mathbb{R}^n$ *with the standard inner product, and let* $\varphi_n(x) = [x^\top, 0]^\top$ *correspond to padding a vector with a zero. This is a consistent sequence for many standard sequences*

[1] Our theory can be extended to reductive groups.

[2] Formally, we obtain such inclusions inside the direct limit of the sequence.

[3] Formally, the space of freely-described elements is the inverse limit $\varprojlim_n \mathbb{V}_n^{\mathsf{G}_n}$.

$\{\mathsf{G}_n\}$ of groups, including the groups of $n \times n$ orthogonal matrices, and (signed) permutation matrices. Here G_n is embedded in G_{n+1} by sending $g \in \mathsf{G}_n$ represented as an $n \times n$ matrix to $\mathrm{blkdiag}(g, 1)$. For each of these sequences of groups, we get a consistent sequence $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$. When $\mathsf{G}_n = \mathsf{S}_n$ is the group of permutations on n letters, the space of freely-described elements in $\mathscr{V}$ is one-dimensional and consists of sequences $\{\alpha \mathbb{1}_n\}$ for $\alpha \in \mathbb{R}$. The space of freely-described linear maps from $\mathbb{V}_n$ to itself is two-dimensional, and consists of sequences $\{\alpha_1 \mathbb{1}_n \mathbb{1}_n^\top + \alpha_2 I_n\}$ for $\alpha \in \mathbb{R}^2$.

Having formalized freely-described vectors and linear maps, we are ready to define free descriptions of convex sets. These are just sequences of conic descriptions specified using freely-described elements.

Definition 1.6 (Free conic descriptions)**.** *Let $\mathscr{V} = \{\mathbb{V}_n\}$, $\mathscr{W} = \{\mathbb{W}_n\}$, $\mathscr{U} = \{\mathbb{U}_n\}$ be consistent sequences of $\{\mathsf{G}_n\}$-representations, and $\{\mathcal{K}_n \subseteq \mathbb{U}_n\}$ be a sequence of convex cones. A sequence of conic descriptions*

$$\mathcal{C}_n = \{x \in \mathbb{V}_n : \exists y \in \mathbb{W}_n \ \textit{s.t.}\ A_n x + B_n y + u_n \in \mathcal{K}_n\}, \tag{ConicSeq}$$

is called free *if $\{A_n\}, \{B_n\}$, and $\{u_n\}$ are freely-described elements of the consistent sequences $\mathscr{V} \otimes \mathscr{U}$, $\mathscr{W} \otimes \mathscr{U}$, and $\mathscr{U}$, respectively.*

All the descriptions in Examples 1.1 and 1.2 become free when the relevant sequences of vector spaces are endowed with natural consistent sequence structure. Moreover, when $\mathscr{V} \otimes \mathscr{U}, \mathscr{W} \otimes \mathscr{U}, \mathscr{U}$ are all finitely-generated, Definition 1.6 yields a finitely-parametrized family of free descriptions, obtained by choosing bases for the spaces of freely-described $\{A_n\}, \{B_n\}, \{u_n\}$ and viewing the coefficients in these bases as the parameters. These parametric families generalize Example 1.2. More broadly, they can be adapted to specific applications via the choice of embeddings and group actions involved in the consistent sequences, which formally relate instances of different sizes and their symmetries.

We can use free descriptions to describe convex *functions* as well as sets, using the many correspondences between convex sets and functions from convex analysis. For example, given a freely-described sequence of convex sets we can consider the sequence of their support and gauge functions, and given a sequence of convex functions we can consider free descriptions for the sequence of their epigraphs [18, §4]. Thus, Definition 1.6 yields finitely-parametrized infinite sequences of convex functions as well.

Example 1.7 (Parametric convex graph invariants)**.** *A convex graph invariant is a convex function over symmetric matrices (viewed as adjacency matrices of weighted graphs) that is invariant under conjugation of its argument by permutation matrices (viewed as relabelling the vertices of the graphs); for example, the max-cut value of a (weighted) graph is a convex graph invariant [19]. These examples and others are defined for graphs of any size, and therefore they correspond to a sequence of convex functions $\{f_n : \mathbb{S}^n \to \mathbb{R}\}$ such that f_n is invariant under conjugation of its argument by permutation matrices for each n.*

Our framework yields parametric families of convex graph invariants as support or gauge functions of parametric freely-described convex sets. Set $\mathbb{V}_n = \mathbb{S}^n$ with the action of $\mathsf{G}_n = \mathsf{S}_n$ by conjugation, since we seek convex subsets of symmetric matrices invariant under this group action. We choose embeddings $\varphi_n : \mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1}$ by padding with a zero row and column, which corresponds to appending isolated vertices to a graph. This yields the consistent sequence $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$.

To obtain convex sets, we also need to choose description spaces and cones. For simplicity, we choose description spaces $\mathbb{W}_n = 0$ and $\mathbb{U}_n = \mathbb{V}_n$ with the same embeddings and group actions as for $\mathscr{V}$, and the positive semidefinite cones $\mathcal{K}_n = \mathbb{S}^n_+$. Once the relevant consistent sequences and cones have been chosen, the parametric family of freely-described sets, parametrized in this case by $\alpha \in \mathbb{R}^{11}$, are obtained transparently from our framework:

$$\begin{aligned}\mathcal{C}_n = \Bigg\{ X \in \mathbb{S}^n : \alpha_1(\mathbb{1}_n^\top X \mathbb{1}_n)\mathbb{1}_n\mathbb{1}_n^\top + \alpha_2(\mathbb{1}_n^\top X \mathbb{1}_n) I_n + \alpha_3 \mathrm{Tr}(X)\mathbb{1}_n\mathbb{1}_n^\top + \alpha_4 \mathrm{Tr}(X) I_n + \alpha_5\Big(X\mathbb{1}_n\mathbb{1}_n^\top + \mathbb{1}_n\mathbb{1}_n^\top X\Big) \\ + \alpha_6\Big(\mathrm{diag}(X)\mathbb{1}_n^\top + \mathbb{1}_n \mathrm{diag}(X)^\top\Big) + \alpha_7 X + \alpha_8 \mathrm{diag}(X\mathbb{1}_n) + \alpha_9 I \odot X + \alpha_{10}\mathbb{1}_n\mathbb{1}_n^\top + \alpha_{11} I_n \succeq 0 \Bigg\}.\end{aligned} \tag{1}$$

We can obtain a larger parametric family by enlarging the description space. For example, taking k copies of the description spaces and cones above yields convex sets described by k linear matrix inequalities of the form (1), depending on $11k$ parameters.

1.1.2 Structural Aspects of Free Descriptions

We have defined freely-described convex sets in Definition 1.6 by relating the *descriptions* of the convex sets in a sequence $\{\mathcal{C}_n\}$ across dimensions. However, in many applications it is further desirable to directly relate the sets $\mathcal{C}_n$ themselves or the functions that arise from them across dimensions. We study the following two relations (which are dual to each other by [18, Cor. 16.3.2]).

Definition 1.8 (Compatibility conditions)**.** *Let $\{\mathbb{V}_n\}$ be a nested sequence of vector spaces and let $\{\mathcal{C}_n \subseteq \mathbb{V}_n\}$ be a sequence of convex sets (respectively, let $\{f_n : \mathbb{V}_n \to \mathbb{R} \cup \{\infty\}\}$ be a sequence of convex functions). We say that $\{\mathcal{C}_n\}$ (resp., $\{f_n\}$) satisfies*

Intersection compatibility *if $\mathcal{C}_{n+1} \cap \mathbb{V}_n = \mathcal{C}_n$ (resp., $f_{n+1}|_{\mathbb{V}_n} = f_n$);*

Projection compatibility *if $\mathcal{P}_{\mathbb{V}_n}\mathcal{C}_{n+1} = \mathcal{C}_n$ (resp., $\mathcal{P}_{\mathbb{V}_n} f_{n+1} = f_n$).*

Here $(\mathcal{P}_{\mathbb{V}_n} f_{n+1})(x) = \inf_{x' \in \mathcal{P}_{\mathbb{V}_n}^{-1}(x)} f_{n+1}(x')$ defines a convex function on $\mathbb{V}_n$ [18, Thm. 5.7]. One can check that a sequence of convex functions is intersection or projection compatible if and only if its sequence of epigraphs is correspondingly compatible, and a similar correspondence holds for gauge and support functions of compatible sequences of sets; see Section 1.3. In what follows, the nested sequence of spaces in Definition 1.8 will always be a consistent sequence.

The conditions in Definition 1.8 are natural in a variety of applications. For example, many graph parameters remain unchanged when an isolated vertex is appended to the graph, such as the max cut value; hence these are intersection-compatible with respect to the embeddings in Example 1.7. Other parameters, such as the stability number, are nonincreasing when taking induced subgraphs, and any small graph occurs as an induced subgraph of a larger one that has the same parameter value. Hence such parameters are projection-compatible with respect to the same embeddings. In inverse problems, it is desirable to use regularizers that are both intersection and projection compatible. Indeed, if $\{\mathbb{V}_n\}$ is a consistent sequence and we are given data about a signal $x \in \mathbb{V}_N$ that only depends on its projection $\mathcal{P}_{\mathbb{V}_n}x$ onto $\mathbb{V}_n$ for $n < N$, then compatibility of the regularizer ensures that the recovered signal also lies in $\mathbb{V}_n$, see Section 4.8. Compatibility plays a central role in the analysis of approximations for the copositive cones, and the failure of a variant of intersection compatibility for the natural sums-of-squares relaxation of copositive cones underlies several results in this area [20, 21]. Compatibility conditions also arise in noncommutative geometry, where matrix-convex sets are defined as sequences of sets of matrices of each size related across dimensions by conditions stronger, in general, than Definition 1.8 (see Proposition 4.5); in particular, the free spectrahedra in Example 1.2 satisfy both intersection and projection compatibility.

A freely-described sequence of convex sets need not satisfy either intersection or projection compatibility; see Example 3.1 below. Therefore, we investigate freely-described convex sets that additionally satisfy compatibility conditions, and we obtain three structural results pertaining to such sequences of sets. Each of these results is obtained by combining the relevant concepts from representation stability along with notions from convex analysis. We now describe these three results in more detail.

Our first structural result gives conditions under which free descriptions *certify* compatibility, i.e., under which the sequence of sets derived from these descriptions is evidently compatible. Consider a freely-described sequence $\{\mathcal{C}_n\}$ of convex sets (ConicSeq). Assuming that the sequence of cones $\{\mathcal{K}_n\}$ underlying the description of $\{\mathcal{C}_n\}$ is both intersection and projection compatible, we prove in Proposition 3.2 that the sequence of convex sets $\{\mathcal{C}_n\}$ satisfies compatibility conditions if the underlying freely-described elements in (ConicSeq) lie in a convex cone. This result serves as the foundation for our subsequent developments—we use it to show in Section 4 that many sets arising in applications naturally admit free descriptions certifying their compatibility. We also use this result in Section 6 to design an algorithm fitting a freely-described and compatible sequence of sets to data.

Our second structural result is central to the development of our computational framework in Section 6 and it addresses the following question. Given a convex set with a conic description (Conic) in a fixed dimension n_0, when does this description *extend* to a free description of a sequence of sets satisfying compatibility conditions? As a concrete example, suppose we are given a convex relaxation for a combinatorial optimization problem, or a convex regularizer for an inverse problem, in a particular problem dimension; under what conditions can these be extended to problems in any desired dimension? Leveraging our preceding result along with properties of presentation degrees of consistent sequences, we give conditions in Theorem 3.5 on the dimension n_0 and the conic description in that dimension ensuring the desired extendability. We use this result in Section 6 to computationally extend a set fitted to data in a fixed dimension to any other dimension.

Finally, our third structural result develops a notion of a limiting object for a freely-described convex set. Given a consistent sequence $\{\mathbb{V}_n\}$ of G_n-representations, consider the vector space $\mathbb{V}_\infty = \cup_n \mathbb{V}_n$ viewed as a representation of[4] $\mathsf{G}_\infty = \cup_n \mathsf{G}_n$ and let $\overline{\mathbb{V}_\infty}$ denote the completion of $\mathbb{V}_\infty$ with respect to some norm. Consider now a freely-described sequence of convex sets $\{\mathcal{C}_n\}$ that is intersection- and projection-compatible. The set $\mathcal{C}_\infty = \cup_n \mathcal{C}_n$ is a convex subset of $\mathbb{V}_\infty$, and we would like to describe its continuous limit $\overline{\mathcal{C}_\infty}$ inside $\overline{\mathbb{V}_\infty}$. This is natural in many applications where problems of different sizes can be naturally viewed as suitable finite-dimensional discretizations of infinite-dimensional problems. Examples include finite graphs obtained via discretization of continuous graphons, vectors obtained via discretizations of continuous-time signals, and matrices obtained via discretizations of operators between infinite dimensional spaces. We show in Theorem 3.6 that if the free description underlying the sequence $\{\mathcal{C}_n\}$ certifies our compatibility conditions (in the sense of Proposition 3.2) and if the freely-described elements constituting the description extend continuously to their respective limits, then the free description extends to a description of a dense subset of $\overline{\mathcal{C}_\infty}$. In particular, this result yields an infinite-dimensional conic program for optimizing a continuous linear functional over $\overline{\mathcal{C}_\infty}$.

Example 1.9 (Convex graphon parameters)**.** *Consider the freely-described sequence in Example 1.2(c), which is intersection- and projection-compatible with respect to the graphon consistent sequence (see Section 4.7). The elements of $\mathbb{V}_\infty$ are step functions on $[0,1]^2$ known as step graphons, and we endow them with the L_∞ norm as is common in the literature [22]. The completion $\overline{\mathbb{V}_\infty}$ then consists of certain piecewise-continuous graphons, and Theorem 3.6 shows that the limit of the sequence $\{\mathcal{C}_n\}$ in Example 1.2(c) with $L_7 = 0$ is*

$$\overline{\mathcal{C}_\infty} = \Bigg\{ W \in \overline{\mathbb{V}_\infty} : 0 \preceq \Big[(L_1)_{i,j} \int_{[0,1]^2} W(s,t)\, ds\, dt + (L_2)_{i,j} \int_{[0,1]} W(t,t)\, dt + (L_3)_{i,j} \int_{[0,1]} [W(x,t) + W(t,y)]\, dt + (L_4)_{i,j}[W(x,x) + W(y,y)] + (L_5)_{i,j} W(x,y) \Big]_{i,j=1}^{k} + L_6 \Bigg\}.$$

Here positive-semidefiniteness of a matrix-valued function on $[0,1]^2$ is meant in the sense of matrix-valued kernels [23], see (21) for a definition. We require $L_7 = 0$ because the corresponding term does not extend continuously with respect to the L_∞ norm. We derive this limiting description in Proposition 4.7.

We also apply Theorem 3.6 to permutahedra and Schur-Horn orbitopes in Section 4.3 to obtain descriptions of limits in analogy to Example 1.9. The resulting infinite-dimensional conic descriptions yield an infinite-dimensional generalization of the Schur-Horn theorem, which we give in Proposition 4.3. In fact, we obtain limiting descriptions and a Schur-Horn theorem more generally in approximately finite-dimensional (AF) algebras in Appendix C.

1.1.3 Invariant Conic Programs

Our framework yields structural results not only for descriptions of convex sets but also for sequences of optimization problems over them. Specifically, there is a literature showing that many sequences of invariant conic programs indexed by dimension can be solved in constant time. We use representation stability to unify and generalize these results. Such sequences of programs arise in several applications including extremal combinatorics [3, 4] and quantum information [5, 6], as we alluded to previously. Concretely, the programs that arise in certifying homomorphism density inequalities over graphs are invariant under symmetric groups of increasing sizes that relabel the vertices of the graphs involved. Similarly, optimizing traces of matrix polynomials yields programs that are invariant under the unitary or orthogonal groups of increasing sizes that conjugate the matrices involved. We now recall symmetry reductions of invariant programs and explain how constant-sized reductions arise from representation stability.

[4]Formally, these are the direct limits $\varinjlim \mathbb{V}_n$ and $\varinjlim \mathsf{G}_n$.

Consider optimizing a linear functional over a convex subset $\mathcal{C} \subseteq \mathbb{V}$ specified as an affine section of a convex cone $\mathcal{K}$ as in (Conic). If the linear functional, the affine section, and the cone are invariant under the action of a group G, then we can further restrict the constraint set to the invariant subspace $\mathbb{V}^{\mathsf{G}}$, thereby reducing the size of the program [24, §3]. Consider now a freely-described sequence of convex subsets $\{\mathcal{C}_n\}$ of a consistent sequence $\{\mathbb{V}_n\}$ of $\{\mathsf{G}_n\}$-representations given by (ConicSeq). The vectors and linear maps in (ConicSeq) are G_n-invariant as they are given by freely-described elements; if in addition the cones $\mathcal{K}_n$ are G_n-invariant, then the convex sets $\mathcal{C}_n$ are also G_n-invariant. When $\{\mathbb{V}_n\}$ is finitely-generated, so that the spaces of invariants in the sequence are all eventually isomorphic, optimizing a G_n-invariant linear functional over $\mathcal{C}_n$ for sufficiently large n reduces as above to optimization over these constant-dimensional spaces of invariants. However, even though the dimensionality of the variables in the symmetry-reduced programs stabilize, the complexity of the constraints might still grow with n.

To establish that the complexity of the constraints also stabilizes with n, we show that the invariant sections $\{\mathcal{K}_n^{\mathsf{G}_n}\}$ of the cones have a constant-sized description in the following precise sense.

Definition 1.10 (Constant-sized descriptions)**.** *Let $\{\mathbb{U}_n\}$ be a sequence of $\{\mathsf{G}_n\}$ representations and $\{\mathcal{K}_n \subseteq \mathbb{U}_n\}$ a sequence of convex cones. For $t \in \mathbb{N}$, we say that the sequence $\{\mathcal{K}_n^{\mathsf{G}_n} \subseteq \mathbb{U}_n^{\mathsf{G}_n}\}$ admits a* constant-sized description *for $n \geq t$ if there exists a single vector space $\mathbb{U}$ containing a cone $\mathcal{K}$, linear maps $T_n \colon \mathbb{U} \to \mathbb{U}_n^{\mathsf{G}_n}$, and subspaces $\mathbb{L}_n \subseteq \mathbb{U}$ such that $\mathcal{K}_n^{\mathsf{G}_n} = T_n(\mathcal{K} \cap \mathbb{L}_n)$ for all $n \geq t$.*

Note that Definition 1.10 does not require $\{\mathbb{U}_n\}$ to be a consistent sequence, nor in particular that $\mathbb{U}_n$ is embedded into $\mathbb{U}_{n+1}$. Proofs of constant-sized symmetry reductions in the literature have implicitly proceeded in a case-by-case manner by showing that the relevant cones, including symmetric PSD and relative entropy cones, have a constant-sized description in the sense of our Definition 1.10, see [25, 3, 26, 27]. In Section 5, we explain how these constant-sized descriptions can be generalized and derived systematically from an interplay between representation stability and the structure of the cones in question. To that end, we use a stronger form of representation stability known as uniform representation stability, which shows that the whole decomposition into irreducibles of the sequences of representations involved stabilize, rather than only their spaces of invariants; see Section 2.5. Our approach allows us to prove the following results.

Theorem 1.11 (Informal)**.** *Consider the consistent sequence $\mathscr{V}_0 = \{\mathbb{R}^n\}$ with embeddings by zero-padding and the standard actions of one of the sequences of classical Weyl groups as in Example 1.5. For any consistent sequence $\mathscr{V} = \{\mathbb{V}_n\}$ obtained from $\mathscr{V}_0$ by taking finitely-many direct sums, tensor products, symmetric algebras, or skew-symmetric algebras, the invariant sections of the cones $\{\mathrm{Sym}^2_+(\mathbb{V}_n)\}$ admit constant-sized descriptions for $n \geq d+k$, where d and k are the generation and presentation degrees of $\mathscr{V}$, respectively. These degrees can be bounded using the calculus in Theorem 2.11.*

We refer the reader to Theorem 5.1 for the formal statement and its proof. The sequences of classical Weyl groups are the groups of permutations $\{\mathsf{S}_n\}$, signed permutations $\{\mathsf{B}_n\}$, or even signed permutations $\{\mathsf{D}_n\}$. Here $\mathrm{Sym}^2_+(\mathbb{V})$ denotes the collection of nonnegative quadratic forms on $\mathbb{V}$. For example, Theorem 1.11 shows that the sections of the PSD cones $\{\mathbb{S}^n_+\}$ invariant under the usual action of the classical Weyl groups by conjugation admit constant-sized descriptions. Theorem 1.11 is used to derive constant-sized descriptions for cones of invariant sums-of-squares in Theorem 1.12 below. In the following result, we consider polynomials in $\binom{n}{k}$ variables identified with k-subsets of n letters, and show that the cones of sums of squares of such polynomials modulo any sequence of ideals admit constant-sized descriptions. The proof of the following result is given in Section 5.1.

Theorem 1.12. *Let $\{\mathsf{G}_n\}$ be one of the sequences of classical Weyl groups acting as usual on $\mathbb{R}^n$. Let*

$$\mathcal{I}_n \subseteq \bigoplus_{d\geq 0} \mathrm{Sym}^d\left(\bigwedge\nolimits^k \mathbb{R}^n\right) \cong \mathbb{R}[x_{i_1,\ldots,i_k}]_{1\leq i_1<\cdots<i_k\leq n} =: \mathbb{V}_n$$

be G_n-invariant ideals, and let $\mathbb{U}_n = \mathrm{Sym}^{\leq 2d}(\bigwedge^k \mathbb{R}^n)/\mathcal{I}_n$. Consider the sums-of-squares cones $\mathrm{SOS}_{\mathbb{U}_n} = \{f \in \mathbb{U}_n : f \text{ is a sum of squares mod } \mathcal{I}_n\}$. Then the sequence $\{\mathrm{SOS}_{\mathbb{U}_n}^{\mathsf{G}_n}\}$ admits a constant-sized description for $n \geq 2kd$ if $\mathsf{G}_n = \mathsf{S}_n$ or B_n, and for $n \geq 2kd+1$ if $\mathsf{G}_n = \mathsf{D}_n$.

If $k = 1$, $\mathcal{I}_n = (0)$, and $\mathsf{G}_n = \mathsf{S}_n$ then we recover [25, Thms. 4.7, 4.10], and when $\mathsf{G}_n = \mathsf{B}_n$ or D_n we recover [26, Cor. 3.23]. If $k \geq 2$, if $\mathcal{I}_n = (x_I - x_I^2)_{I\subseteq\binom{[n]}{k}}$ is the ideal generated by $x_I^2 - x_I$ where I ranges over all k-subsets of $[n]$, and if $\mathsf{G}_n = \mathsf{S}_n$, we recover [3, Thm. 2.4]. Theorem 1.12 generalizes all of these results to include any of the classical Weyl groups and any sequence of invariant ideals.

An application of Theorem 1.12 is obtaining constant-sized SDPs to certify graph homomorphism density inequalities [3, 4]. Many problems in extremal combinatorics can be recast as proving polynomial inequalities between homomorphism densities of graphs, which is the fraction of maps between the vertex sets of two graphs that define graph homomorphisms. A simple example is Mantel's theorem, which states that the maximum number of edges in a triangle-free graph is $\lfloor n^2/4 \rfloor$. Razborov proposed a method of certifying such inequalities using *flag algebras* [28], which were shown in [3, 4] to be sums-of-squares certificates of certain symmetric polynomial inequalities. Razborov's flags are interpreted in [3, §3] as "free" spanning sets for spaces of symmetric polynomials. Formally, they are freely-described elements in the sense of Definition 1.4. Our framework shows that both the existence of such freely-described spanning sets and the resulting constant-sized SDPs are consequences of representation stability.

We obtain similar results for invariant sections of cones derived from relative entropy inequalities that have recently played a role in polynomial optimization [29, 30]. Our main result for these cones is stated in Theorem 5.6, which generalizes [27, Thm. 5.3] from permutation groups to the other classical Weyl groups.

1.1.4 Free Convex Regression

As our final contribution, we apply our framework to develop an algorithm for a free analog of the convex regression problem. In the classic setting of this problem, the objective is to fit a convex function $f\colon \mathbb{V} \to \mathbb{R}$ to a dataset $\{(x_i, y_i)\}_{i=1}^D$ of inputs $x_i \in \mathbb{V}$ and outputs $y_i \in \mathbb{R}$ such that $f(x_i) \approx y_i$. In our free extension of this problem, we have a sequence $\{\mathbb{V}_n\}$ of vector spaces, usually of growing dimension, and data $\{(x_i, y_i) \in \mathbb{V}_{n_i} \oplus \mathbb{R}\}$ in finitely-many dimensions n_i. We then seek an *infinite* sequence of convex functions $\{f_n\colon \mathbb{V}_n \to \mathbb{R}\}$ that can be instantiated in any dimension, satisfying $f_{n_i}(x_i) \approx y_i$ in the dimensions in which data is available.

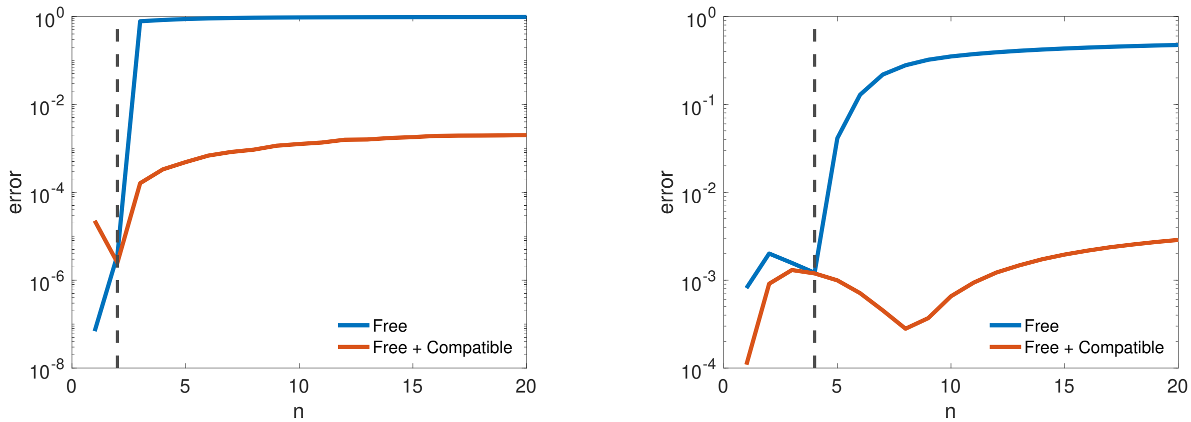
We address this problem by leveraging the framework described above to fit a freely-described convex function to the given data. Concretely, we begin by endowing the vector spaces $\{\mathbb{V}_n\}$ containing the training data with the structure of a consistent sequence based on the symmetries of the problem at hand and the relations between problem instances in different dimensions. We then select description spaces $\{\mathbb{W}_n\}, \{\mathbb{U}_n\}$ and cones $\{\mathcal{K}_n \subseteq \mathbb{U}_n\}$ with respect to which our convex functions are to be described as in (ConicSeq), with freely-described vectors and linear maps parametrizing the desired family of freely-described functions; richer description spaces generally yield more expressive families of convex functions, but fitting a function from such a family is more expensive and requires data in higher dimensions by Theorem 3.5.

Having chosen description spaces and cones as above, we have completely specified our family of freely-described functions and turn to the problem of fitting an element of this family to data. We do so in a fully algorithmic fashion, without needing to write down an explicit formula for members of our parametric family, using the following three steps. First, we compute a basis for invariant vectors and linear maps in the dimensions in which we have data using the algorithm of [31]. Second, we numerically identify coefficients in this basis that fit the data using an alternating minimization procedure based on convex duality. Third, we extend the invariant vectors and linear maps we identified in the data dimensions to other dimensions by solving linear systems arising from Definition 1.4. Our procedure is summarized in Algorithm 1 and detailed in Section 6, and our implementation is publicly available at `https://github.com/eitangl/anyDimCvxSets`. To summarize, once the choice of consistent sequence structure and description spaces is made based on the structure underlying an application, the dimensions of the available data, and the desired richness of the family of functions, the remainder of the procedure is fully computational.

We demonstrate our approach by obtaining semidefinite approximations of two non-semidefinite representable functions: the ℓ_π norm $\|x\|_\pi = \left(\sum_i |x_i|^\pi\right)^{1/\pi}$, and the following (nonnegative and positively homogeneous) variant of the quantum entropy function:

$$f(X) = \mathrm{Tr}[(X + \mathrm{Tr}(X) I) \log(X/\mathrm{Tr}(X) + I)]. \tag{2}$$

The ℓ_π norm of a vector is defined for vectors of any length, is invariant under signed permutations, remains unchanged under zero-padding of its input, and can only increase if we append any other entry to a given



(a) Learning ℓ_π.

(b) Learning quantum entropy.

Figure 1: Errors for learning the ℓ_π norm and the quantum entropy variant (2). The dashed vertical lines denote the max n for which data is available.

input. It is therefore both intersection and projection compatible with respect to the embeddings and projections of Example 1.5. The function (2) is well-defined for (positive-semidefinite) inputs of all sizes, is invariant under conjugation by orthogonal matrices, and its value is unchanged if we zero-pad its input. Thus, it is intersection-compatible on $\{\mathbb{S}^n\}$ with embeddings given by by zero-padding. Given evaluations of these two functions on low-dimensional inputs, we use our procedure to fit semidefinite approximations to them by choosing sequences of PSD cones for our descriptions, and fitting a freely-described and compatible sequence of convex functions. Thus, we ensure that our approximations are also group-invariant and satisfy the same compatibility properties as the underlying ground-truth functions (we detail our description spaces in Section 6.3)). Figure 1 shows the error in our semidefinite approximations for different dimensions when we search over all freely-described sets in our family when fitting to data, and when we search only over the smaller family of compatible sequences. We see that imposing compatibility ensures that the error increases gracefully when extending to higher dimensions.

Obtaining semidefinite approximations for quantum information theoretic quantities such as (2) facilitates the use of semidefinite programming solvers in convex optimization problems involving quantum entropy [32, 33, 34, 35]. We mention that the authors of [36] derived semidefinite approximations of the quantum entropy and related functions from an analytic perspective. We show in Section 4.5 that their description is free but does not certify compatibility, in contrast to the description we obtain from data using the preceding procedure.

1.2 Related Work

We briefly survey several related areas.

Extended formulations of convex sets: There is a large literature as well as a systematic framework on extended formulations in which conic descriptions of a convex set in a fixed dimension are investigated; see [37] for a review. The goal in this body of work is to express 'complicated' convex sets in $\mathbb{R}^d$ (e.g., polyhedra with many facets) as linear images of affine sections of 'simple' convex cones in a space that is not much larger than d. This framework is also applied to study *equivariant lifts* of group-invariant convex sets, which are descriptions of the form (Conic) consisting of group-invariant cones, vectors, and linear maps, i.e., conic descriptions that make evident the group invariance of the convex set; see [38] and [37, §4.3]. These are precisely the type of descriptions we consider for each of the convex sets in our sequences. Moreover, while the literature on extended formulations is typically articulated in the setting of a convex set in a *fixed* dimension, many results in the area implicitly concern descriptions of a *sequence* $\{\mathcal{C}_n\}$ of convex sets and the complexity of these descriptions as a function of n [39, 38, 40, 41]. Thus, there are several points of contact with the present paper. In fact, many (though not all) descriptions proposed in the literature on extended formulations can in fact be instantiated in any dimension, and are moreover free in our sense as we show in Section 4. More broadly, to the best of our knowledge, descriptions of convex sets that arise from a systematic consideration of relations between dimensions have not been studied previously. By bringing such considerations to the fore, the present effort elucidates the representation-theoretic phenomena and their interaction with the convex-geometric aspects underpinning convex sets that can be instantiated in any dimension.

Free spectrahedra, noncommutative convex algebraic geometry: A broad research program pursued in several areas involves the study of "matrix" or noncommutative analogues of classical "scalar" or commutative objects. Examples include random matrix theory and free probability [42] in which matrix-valued random variables and their limits are the object of study as opposed to scalar-valued ones; and noncommutative algebraic geometry [13], in which polynomials in noncommuting variables and their evaluations on matrices are studied as opposed to standard polynomials in commuting variables that are evaluated on scalars [43, 44, 45, 46]. Applying this program to convex sets yields matrix-convex sets and free spectrahedra as in Example 1.2(b). We refer the reader to [13, 14] for surveys and [47, 48] for some applications. In analogy to the present paper, results in this area explicitly pertain to sequences of sets which are "freely-described", in the sense that their description can be instantiated in any dimension. For example, free spectrahedra are sequences of sets described by a single linear matrix inequality, and free algebraic varieties are defined by the same noncommutative polynomials instantiated on matrices of any size. Another point of contact with our work is the consideration of relations between the sets in the sequence across dimensions, such as matrix-convex combinations which have been formalized and studied in this literature. Our notion of free descriptions is more general than the ones in this literature however, and it allows us to derive more flexible families of freely-described sets which are adapted to the structure underlying a broader range of applications. Further, the relations between sets in different dimensions we consider in this paper are less restrictive than matrix convexity, and they yield more general families of sets than free spectrahedra (free spectrahedra may be obtained in our framework via particular instantiations of description spaces, see Section 4.4).

Representation stability: Representation stability arose out of the observation that the decomposition into irreducibles stabilizies for many sequences of representations. This phenomenon has been formalized in [16] using consistent sequences, and it has been further studied in [49, 50, 51] from a categorical perspective and in [52, 53, 54] from a limits-based perspective. We relate the categorical and limits-based formalisms to our setting in appendix A and Section 2.7, respectively, and we refer the reader to [55, 56, 57] for introductions to this area.

Representation stability has been used to study sequences of polyhedral cones and their infinite-dimensional limits [58], as well as sequences of algebraic varieties, their defining equations, and their infinite-dimensional limits [59, 60, 61]. An important distinction between these works and ours is our application of representation stability to *descriptions* of convex sets rather than to their extreme points or rays as in [58]. Thus, we are able to study non-polyhedral sets such as spectrahedra and sets defined by relative entropy programs. Similarly, our study of infinite-dimensional limits in Section 3.3 focuses on limiting descriptions and not just on limits of the sets themselves.

1.3 Notation and Basics

We assume familiarity with the basics of representation theory and convex analysis, and we refer the reader to [62, 63] and [18], respectively, for background. In what follows, we review a few basic notions from these areas and introduce notation used throughout the paper. We list several standard groups and constructions involving vector spaces in Table 1.

Basics: We denote $[n] = \{1, \ldots, n\}$. For $i \leq j$ we denote by (i, j) the transposition permuting letters i and j. For real numbers $a < b$, denote $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$. For $i \in [n]$, we denote by $e_i \in \mathbb{R}^n$ the ith standard basis vector with a 1 in the ith entry and zero everywhere else, and we write $e_i^{(n)}$ when we wish to emphasize the dimension. If $x \in \mathbb{R}^n$, we denote by $\mathrm{diag}(x) \in \mathbb{S}^n$ the diagonal matrix with x on the diagonal. If $X \in \mathbb{R}^{n\times n}$, we denote by $\mathrm{diag}(X) \in \mathbb{R}^n$ the vector of its diagonal elements. All vector spaces in this paper are finite-dimensional real vector spaces equipped with an inner product $\langle\cdot,\cdot\rangle$ unless stated otherwise. We

Symmetric group	$\mathsf{S}_n = \{g \in \mathbb{R}^{n\times n} : g \text{ is a permutation matrix.}\}$
Signed symmetric group	$\mathsf{B}_n = \{g \in \mathbb{R}^{n\times n} : g \text{ is a signed permutation matrix}\}$
Even-signed symmetric group	$\mathsf{D}_n = \{g \in B_n : g \text{ flips evenly-many signed}\}$
Orthogonal group	$\mathsf{O}_n = \{g \in \mathbb{R}^{n\times n} : g^\top g = I_n\}$
Space of linear maps	$\mathcal{L}(\mathbb{V},\mathbb{U}) = \{A\colon \mathbb{V}\to\mathbb{U} \text{ linear}\}; \qquad \mathcal{L}(\mathbb{V}) = \mathcal{L}(\mathbb{V},\mathbb{V}).$
Direct sum	$\mathbb{V}\oplus\mathbb{U} = \mathbb{V}\times\mathbb{U} = \{(v,u) : v\in\mathbb{V}, u\in\mathbb{U}\}.$
Direct powers	$\mathbb{V}^k = \mathbb{V}^{\oplus k} = \underbrace{\mathbb{V}\oplus\cdots\oplus\mathbb{V}}_{k \text{ times}}.$
Tensor product	$\mathbb{V}\otimes\mathbb{U} = \operatorname{span}\{v\otimes u : v\in\mathbb{V}, u\in\mathbb{U}\} \cong \mathcal{L}(\mathbb{V},\mathbb{U}).$
Tensor power	$\mathbb{V}^{\otimes k} = \underbrace{\mathbb{V}\otimes\cdots\otimes\mathbb{V}}_{k \text{ times}}.$
Symmetric algebra	$\begin{aligned}\operatorname{Sym}^k(\mathbb{V}) &= \operatorname{span}\{v_1\cdots v_k : v_i\in\mathbb{V}\}\\ &= \{\text{polynomials of degree } = k \text{ on } \mathbb{V}\}\\ &= \{\text{symmetric tensors of order } k \text{ over } \mathbb{V}\}\\ \operatorname{Sym}^{\le k}(\mathbb{V}) &= \bigoplus_{i=0}^k \operatorname{Sym}^i(\mathbb{V}).\end{aligned}$
Alternating algebra	$\begin{aligned}\textstyle\bigwedge^k \mathbb{V} &= \operatorname{span}\{v_1\wedge\cdots\wedge v_k : v_i\in\mathbb{V}\}\\ &= \{\text{skew-symmetric tensors of order } k \text{ over } \mathbb{V}\}.\end{aligned}$
Symmetric matrices	$\mathbb{S}^n = \{X\in\mathbb{R}^{n\times n} : X^\top = X\} = \operatorname{Sym}^2(\mathbb{R}^n).$
Skew-symmetric matrices	$\operatorname{Skew}(n) = \{X\in\mathbb{R}^{n\times n} : X^\top = -X\} = \bigwedge^2\mathbb{R}^n.$
Spaces of invariants	$\mathbb{V}^{\mathsf{G}} = \{v\in\mathbb{V} : g\cdot v = v \text{ for all } g\in\mathsf{G}\},$ $\mathcal{L}(\mathbb{V},\mathbb{U})^{\mathsf{G}} = \{A\in\mathcal{L}(\mathbb{V},\mathbb{U}) : gA = Ag \text{ for all } g\in\mathsf{G}\}.$

Table 1: Commonly-used groups and vector spaces. Here $\mathbb{V},\mathbb{U}$ are finite-dimensional vector spaces.

emphasize that some of the inner products we use are nonstandard, so the transpose of a matrix and the adjoint of the linear operator it represents may differ. Given a subspace $\mathbb{W}\subseteq\mathbb{V}$, we denote by $\mathcal{P}_{\mathbb{W}}\colon\mathbb{V}\to\mathbb{W}$ the orthogonal projection onto $\mathbb{W}$. We denote by $\mathbb{R}^n_+$ the cone of entrywise nonnegative length-n vectors, and by $\mathbb{S}^n_+$ the cone of PSD $n\times n$ matrices. If $\mathbb{V}$ is a vector space, we let $\operatorname{Sym}^2_+(\mathbb{V})\cong\mathbb{S}^{\dim\mathbb{V}}_+$ denote the cone of PSD linear maps in $\mathcal{L}(\mathbb{V})$.

Representation theory: A (linear) action of a group G on a finite-dimensional vector space $\mathbb{V}$ is given by a group homomorphism $\rho\colon\mathsf{G}\to\mathrm{GL}(\mathbb{V})$. Usually ρ is clear from context and we omit it, writing $g\cdot v = \rho(g)v$ for $g\in\mathsf{G}$ and $v\in\mathbb{V}$ instead. All the groups we consider are compact and all group actions are orthogonal, meaning $\langle g\cdot x, g\cdot y\rangle = \langle v, v'\rangle$ for all $x,y\in\mathbb{V}$. The action of G on $\mathbb{V}$ induces an action on $\mathbb{V}^{\otimes k}$ and $\operatorname{Sym}^k(\mathbb{V})$ by setting $g\cdot v_1\otimes\cdots\otimes v_k = (gv_1)\otimes\cdots\otimes(gv_k)$ and $g\cdot(v_1\cdots v_k) = (gv_1)\cdots(gv_k)$ and extending by linearity. If $\mathbb{V}$ and $\mathbb{U}$ are both representations of G, we have an action of G on $\mathbb{V}\otimes\mathbb{U}$ by $g\cdot(v\otimes u) = (gv)\otimes(gu)$ (and extending by linearity) and on $\mathcal{L}(\mathbb{V},\mathbb{U})$ by $g\cdot A = gAg^{-1}$, thus making the representations $\mathbb{V}\otimes\mathbb{U}$ and $\mathcal{L}(\mathbb{V},\mathbb{U})$ isomorphic. Linear maps invariant under this preceding group action are also called *equivariant* or *intertwining*, since they are precisely the linear maps commuting with the group elements. We denote the group ring of G by $\mathbb{R}[\mathsf{G}] = \operatorname{span}\{e_g\}_{g\in\mathsf{G}}$, where e_g is a basis element indexed by the group element g. Note that a representation of G is the same as a module over the ring $\mathbb{R}[\mathsf{G}]$.

If $\mathsf{H} \subseteq \mathsf{G}$ is a subgroup and $\mathbb{V}$ is a representation of H, the *induced representation* of G from $\mathbb{V}$ is $\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(\mathbb{V}) = \mathbb{R}[\mathsf{G}] \otimes_{\mathbb{R}[\mathsf{H}]} \mathbb{V}$. We have $\dim \mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(\mathbb{V}) = |\mathsf{G}/\mathsf{H}| \dim \mathbb{V}$, and we apply this notion only when H has finite index in G. If $g_1 = \mathrm{id}, g_2, \ldots, g_k$ are coset representatives for G/H, we have

$$\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(\mathbb{V}) = \bigoplus_{i=1}^{k} g_i \mathbb{V}, \tag{Ind}$$

together with the following action of G: If $g \in \mathsf{G}$ is (uniquely) written as $gg_i = g_j h$ for some $i, j \in [k]$ and $h \in \mathsf{H}$, then $g \cdot g_i v = g_j(h \cdot v)$ for any $v \in \mathbb{V}$. This construction is independent of the choice of coset representatives.

As vector spaces, we have an isomorphism $\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(\mathbb{V}) \cong \mathbb{V}^{|\mathsf{G}/\mathsf{H}|}$. Hence, an H-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$ induces a G-invariant inner product on $\mathbb{V}^{|\mathsf{G}/\mathsf{H}|}$ by setting $\langle g_i v, g_j u \rangle = \delta_{i,j} \langle v, u \rangle$ for $v, u \in \mathbb{V}$ and $i, j \in [|\mathsf{G}/\mathsf{H}|]$. Here $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. We have an isomorphism $(\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \mathbb{V})^{\mathsf{G}} \cong \mathbb{V}^{\mathsf{H}}$ sending $v \in \mathbb{V}^{\mathsf{H}}$ to $\sum_i g_i v \in (\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \mathbb{V})^{\mathsf{H}}$ and $\widetilde{v} \in (\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \mathbb{V})^{\mathsf{H}}$ to $\mathcal{P}_{\mathbb{V}} \widetilde{v} \in \mathbb{V}^{\mathsf{H}}$.

If $\mathsf{H} \subseteq \mathsf{H}'$ and $\mathsf{G} \subseteq \mathsf{G}'$ such that $\mathsf{H}' \cap \mathsf{G} = \mathsf{H}$, then we have the inclusion $\mathsf{G}/\mathsf{H} \hookrightarrow \mathsf{G}'/\mathsf{H}'$ sending $g\mathsf{H} \mapsto g\mathsf{H}'$; this in turn yields an inclusion $\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \mathbb{V} \hookrightarrow \mathrm{Ind}_{\mathsf{H}'}^{\mathsf{G}'} \mathbb{V}$ between induced representations by completing a set of coset representatives for G/H to representatives for G'/H'. Here $\mathbb{V}$ is assumed to be an H'-representation.

If $\mathbb{V}, \mathbb{U}$ are H-representations and $A \in \mathcal{L}(\mathbb{V}, \mathbb{U})^{\mathsf{H}}$, we can extend A to a map $\mathrm{Ind}(A) \colon \mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \mathbb{V} \to \mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \mathbb{U}$ defined by $\mathrm{Ind}(A)(g_i v) = g_i(Av)$ where g_i is one of the above coset representatives and $v \in \mathbb{V}$. If $\mathbb{V}$ is a G-representation and $\mathbb{W} \subseteq \mathbb{V}$ is an H-subrepresentation, there is a G-equivariant linear map $\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}} \mathbb{W} \to \mathbb{V}$ sending $g \otimes w \mapsto g \cdot w$ whose image is precisely $\mathbb{R}[\mathsf{G}]\mathbb{W} = \mathrm{span}\{g \cdot w\}_{g \in \mathsf{G}, w \in \mathbb{W}}$.

Convex sets and functions: The epigraph of a convex function $f \colon \mathbb{V} \to \mathbb{R} \cup \{\infty\}$ is the convex set $\{(x, t) \in \mathbb{V} \oplus \mathbb{R} : f(x) \leq t\}$. If $\mathcal{C} \subseteq \mathbb{V}$ is a convex set then its gauge function (also called Minkowski functional) is $\gamma_{\mathcal{C}}(x) = \inf\{t : x \in t\mathcal{C}\}$ and its support function is $h_{\mathcal{C}}(x) = \sup\{\langle y, x \rangle : y \in \mathcal{C}\}$.

Our compatibility conditions for convex sets in Definition 1.8 imply compatibility for convex functions derived from them using the above correspondences, and vice versa. Indeed, it is easy to check that a sequence of convex functions is intersection (resp., projection) compatible if and only if the sequence of their epigraphs is intersection (projection) compatible. Similarly, if a sequence of convex sets is intersection (resp., projection) compatible, then the sequence of their gauge functions is intersection (resp., projection) compatible. The correspondences between compatibility conditions between sets and their support functions is a bit subtler. If a sequence of sets is projection-compatible, then the sequence of their support functions is intersection-compatible. If a sequence of *compact* sets is intersection-compatible, then their support functions are projection-compatible.

2 Background on Representation Stability

We review some fundamental definitions and results from the representation stability literature, which studies consistent sequences $\{\mathbb{V}_n\}$ of $\{\mathsf{G}_n\}$-representations as in Definition 1.3. We further require a notion of maps between consistent sequences, which enables us to define embeddings, quotients, and isomorphisms of consistent sequences.

Definition 2.1 (Morphisms of sequences). *If $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$ and $\mathscr{U} = \{(\mathbb{U}_n, \psi_n)\}$ are two consistent sequences of $\{\mathsf{G}_n\}$-representations, then a* morphism of consistent sequences *$\mathscr{A} \colon \mathscr{V} \to \mathscr{U}$ is a collection of linear maps $\mathscr{A} = \{A_n \colon \mathbb{V}_n \to \mathbb{U}_n\}$ such that the following hold for each n:*

(a) A_n is G_n-equivariant;

(b) $A_{n+1}\varphi_n = \psi_n A_n$.

If φ_n and ψ_n are inclusions, condition (b) above becomes $A_{n+1}|_{\mathbb{V}_n} = A_n$. Note that morphisms are freely-described elements (as in Definition 1.4) of $\mathscr{V} \otimes \mathscr{U}$, but the converse may not hold. Morphisms of sequences have appeared in the representation stability literature as the natural notion of maps between sequences, see [49, Def. 2.1.1] and [50, §3.2].

2.1 Generation Degree

To relate invariants and equivariants across dimensions, we need canonical isomorphisms between spaces of invariants in a consistent sequence. Proposition 2.3 below shows that the projections $\mathcal{P}_{\mathbb{V}_n}$ are such isomorphisms, using the following parameter introduced in [49] to control the complexity of a consistent sequence.

Definition 2.2 (Generation degree)**.** *A consistent sequence $\mathscr{V} = \{\mathbb{V}_n\}$ of $\{\mathsf{G}_n\}$-representations is* generated in degree d *if $\mathbb{R}[\mathsf{G}_n]\mathbb{V}_d = \mathbb{V}_n$ for all $n \geq d$. The smallest d for which this holds is called the* generation degree *of the sequence. A subset $\mathcal{S} \subseteq \mathbb{V}_d$ is called a* set of generators *for $\mathscr{V}$ if $\mathbb{R}[\mathsf{G}_n]\mathcal{S} = \mathbb{V}_n$ for all $n \geq d$. A sequence is* finitely-generated *if it is generated in degree d for some $d < \infty$.*

Note that $\mathbb{R}[\mathsf{G}_n]\mathcal{S} = \mathrm{span}\{gx\}_{g\in\mathsf{G}_n, x\in\mathcal{S}}$, so that $\mathscr{V}$ is generated in degree d if the span of the G_n-orbit of $\mathbb{V}_d$, when embedded in $\mathbb{V}_n$, is all of $\mathbb{V}_n$ for any $n \geq d$. Note also that if $\mathscr{V}$ is generated in degree d then $\mathbb{V}_d$ is a set of generators for $\mathscr{V}$.

Proposition 2.3. *Suppose $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$ is a consistent sequence of $\{\mathsf{G}_n\}$-representations generated in degree d. Then the restrictions of the projections $\varphi_n^*|_{\mathbb{V}_{n+1}^{\mathsf{G}_{n+1}}} : \mathbb{V}_{n+1}^{\mathsf{G}_{n+1}} \to \mathbb{V}_n^{\mathsf{G}_n}$ to spaces of invariants are injective for all $n \geq d$, and are therefore isomorphisms for all large enough n.*

Proof. First, the map φ_n^* is G_n-equivariant because G_n acts orthogonally and $\mathsf{G}_n \subseteq \mathsf{G}_{n+1}$, so it maps G_{n+1}-invariants in $\mathbb{V}_{n+1}$ to G_n-invariants in $\mathbb{V}_n$. Second, suppose $\varphi_n^*(v) = 0$ for some $v \in \mathbb{V}_{n+1}^{\mathsf{G}_{n+1}}$ with $n \geq d$. For any $u \in \mathbb{V}_{n+1}$, write $u = \sum_i g_i\varphi_n(u_i)$ where $u_i \in \mathbb{V}_n$ and $g_i \in \mathsf{G}_{n+1}$. Because v is G_{n+1}-invariant, we have $\langle v, u\rangle = \langle \varphi_n^*(v), \sum_i u_i\rangle = 0$. As $u \in \mathbb{V}_{n+1}$ was arbitrary, we conclude that $v = 0$. Thus, φ_n^* maps $\mathbb{V}_{n+1}^{\mathsf{G}_{n+1}}$ injectively into $\mathbb{V}_n^{\mathsf{G}_n}$, so that $\dim \mathbb{V}_n^{\mathsf{G}_n} \geq \dim \mathbb{V}_{n+1}^{\mathsf{G}_{n+1}}$, for all $n \geq d$. Therefore, the sequence of dimensions $\dim \mathbb{V}_n^{\mathsf{G}_n}$ eventually stabilizes, at which point φ_n^* becomes an isomorphism. ☐

Note that $\varphi_n^* = \mathcal{P}_{\mathbb{V}_n}$ is precisely the orthogonal projection onto $\mathbb{V}_n$. Proposition 2.3 is stated in the representation stability literature in terms of the adjoints of the projections, viewed as maps between *co*invariants, see [49, §3] for example.

2.2 Presentation Degree

While boundedness of the generation degree of a consistent sequence ensures that the projections eventually become isomorphisms, providing a precise quantification of this phenomenon requires a more sophisticated concept, namely the *presentation degree*. We describe this concept after giving some preliminary definitions. Our presentation here is brief, and we refer the reader to Appendix B for a more detailed derivation of these notions motivated by our computational considerations.

Definition 2.4 (Centralizing subgroups)**.** *Let $\mathscr{V} = \{\mathbb{V}_n\}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations. For any $d \leq n$, define the* centralizing subgroups *of $\mathbb{V}_d$ by*

$$\mathsf{H}_{n,d} = \{g \in \mathsf{G}_n : g \cdot v = v \text{ for all } v \in \mathbb{V}_d\}.$$

Note that the subgroup generated by G_d and $\mathsf{H}_{n,d}$ inside G_n is the set of products $\mathsf{G}_d\mathsf{H}_{n,d} = \{gh : g \in \mathsf{G}_d,\ h \in \mathsf{H}_{n,d}\}$ since $ghg^{-1} \in \mathsf{H}_{n,d}$ for all $g \in \mathsf{G}_d$ and $h \in \mathsf{H}_{n,d}$.

Definition 2.5 ($\mathscr{V}$-modules)**.** *Let $\mathscr{V} = \{\mathbb{V}_n\}$ and $\mathscr{U} = \{\mathbb{U}_n\}$ be consistent sequences of $\{\mathsf{G}_n\}$-representations, and let $\{\mathsf{H}_{n,d}\}_{d\leq n}$ be the centralizing subgroups of $\mathscr{V}$ as in Definition 2.4. We say that $\mathscr{U}$ is a* $\mathscr{V}$-module *if $\mathbb{U}_d \subseteq \mathbb{U}_n^{\mathsf{H}_{n,d}}$ for all $d \leq n$.*

Definition 2.6 (Induction and algebraically free[5] sequences)**.** *Let $\mathscr{V}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations, and for $d \leq n$ let $\mathsf{H}_{n,d} \subseteq \mathsf{G}_n$ be its centralizing subgroups.*

[5] Freeness here is meant in the algebraic sense of being generated by generators with no nontrivial relations between them (see Appendix B), in contrast to Definition 1.6 where it is meant in the sense of dimension-free descriptions.

(a) *Fix $d \in \mathbb{N}$ and a G_d-representation $\mathbb{W}$ on which $\mathsf{H}_{d,d}$ acts trivially. For each $n \geq d$, we view $\mathbb{W}$ as a $\mathsf{G}_d\mathsf{H}_{n,d}$-representation on which $\mathsf{H}_{n,d}$ acts trivially. The associated* $\mathscr{V}$-induction sequence *is defiend as:*

$$\mathrm{Ind}_{\mathsf{G}_d}(\mathbb{W}) = \left\{\mathrm{Ind}_{\mathsf{G}_d\mathsf{H}_{n,d}}^{\mathsf{G}_n}\mathbb{W}\right\}_n,$$

where the induced representation is taken to be 0 when $n < d$. This is a $\mathscr{V}$-module.

(b) *A consistent sequence $\mathscr{F}$ is an* algebraically free $\mathscr{V}$-module *if it is a direct sum of $\mathscr{V}$-induction sequences. The sequence $\mathscr{V}$ itself is* algebraically free *if it is an algebraically free $\mathscr{V}$-module.*

Definition 2.7 (Relation and presentation degrees)**.** *Let $\mathscr{V}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations. We say that a $\mathscr{V}$-module $\mathscr{U}$ is* generated in degree d, related in degree r, and presented in degree $k = \max\{d, r\}$ *if there exists an algebraically free $\mathscr{V}$-module $\mathscr{F}$ generated in degree d, and a surjective morphism of sequences $\mathscr{F} \to \mathscr{U}$ whose kernel is generated in degree r. The smallest k for which this holds is called the* presentation degree *of $\mathscr{U}$.*

Note that the presentation degree is at least as large as the generation degree (cf. Definition 2.2).

Example 2.8. *Let $\mathbb{V}_n = \mathbb{R}^n$ with embeddings by zero-padding as in Example 1.5. Recall that this is a consistent sequence for each of the sequences of groups $\mathsf{G}_n = \mathsf{O}_n, \mathsf{B}_n, \mathsf{D}_n, \mathsf{S}_n$ acting by their standard $n \times n$ matrix representations. Here $\mathsf{H}_{n,d}$ is the subgroup of $n \times n$ orthogonal or signed permutation matrices fixing the first d coordinates.*

This sequence is generated in degree 1 for all the groups listed above. Indeed, any of the canonical basis vectors e_i are obtained from the first one e_1 via the action of S_n.

If $\mathsf{G}_n = \mathsf{B}_n$ or S_n then this sequence is algebraically free and hence presented in degree 1 as well, while if $\mathsf{G}_n = \mathsf{D}_n$ then it is not free but presented in degree 2. Indeed, we have $|\mathsf{D}_n/\mathsf{D}_1\mathsf{H}_{n,1}| = 2n$ when $n \geq 2$ with coset representatives $(1, i)s^p$ for $p \in \{0, 1\}$, $i \in [n]$ where $s = \mathrm{diag}(-1, -1, 1 \ldots, 1)$. Hence (Ind) yields $\mathrm{Ind}_{\mathsf{D}_1\mathsf{H}_{n,1}}^{\mathsf{D}_n}\mathbb{R} = \mathbb{R}^n \oplus \mathbb{R}^n$ on which $\sigma \in \mathsf{S}_n \subseteq \mathsf{D}_n$ acts by $\sigma(x, y) = (\sigma x, \sigma y)$ and $s(x, y) = ([y_1, y_2, x_3, \ldots, x_n]^\top, [x_1, x_2, y_3, \ldots, y_n]^\top)$. We have equivariant linear maps $\mathrm{Ind}_{\mathsf{D}_1\mathsf{H}_{n,1}}^{\mathsf{D}_n}\mathbb{R} \to \mathbb{R}^n$ sending $(x, y) \mapsto x - y$ giving a morphism of sequences $\mathrm{Ind}_{\mathsf{D}_1}\mathbb{R} \to \{\mathbb{R}^n\}$ with kernel generated in degree 2.

The presentation degree enables us to strengthen Proposition 2.3 and to quantify more precisely when the projections there become isomorphisms.

Proposition 2.9. *Let $\mathscr{V}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations and $\mathscr{U}$ be a $\mathscr{V}$-module presented in degree k. Then the maps $\mathcal{P}_{\mathbb{U}_n} : \mathbb{U}_{n+1}^{\mathsf{G}_{n+1}} \to \mathbb{U}_n^{\mathsf{G}_n}$ are isomorphisms for all $n \geq k$.*

Proof. As $\mathscr{U}$ is presented in degree k, there exists an algebraically free $\mathscr{V}$-module $\mathscr{F} = \{\mathbb{F}_n\}$ and a surjective morphism $\mathscr{F} \to \mathscr{U}$ such that both its kernel $\mathscr{K} = \{\mathbb{K}_n\}$ and $\mathscr{F}$ itself are generated in degree k. Because each map $\mathbb{F}_n \to \mathbb{U}_n$ is a G_n-equivariant surjection with kernel $\mathbb{K}_n$, its restriction to invariants $\mathbb{F}_n^{\mathsf{G}_n} \to \mathbb{U}_n^{\mathsf{G}_n}$ is surjective with kernel $\mathbb{K}_n^{\mathsf{G}_n}$.

As $\mathscr{F}$ is an algebraically free $\mathscr{V}$-module, there exist integers d_j and G_{d_j}-representations $\mathbb{W}_{d_j}$ satisfying $\mathscr{F} = \bigoplus_j \mathrm{Ind}_{\mathsf{G}_{d_j}}\mathbb{W}_{d_j}$. Such $\mathscr{F}$ has generation degree $\max_j d_j \leq k$. Therefore, letting $\{\mathsf{H}_{n,d}\}$ be the centralizing subgroups of $\mathscr{V}$, we have for $n \geq k$ (see Section 1.3)

$$\mathbb{F}_n^{\mathsf{G}_n} = \bigoplus_j \left(\mathrm{Ind}_{\mathsf{G}_{d_j}\mathsf{H}_{n,d_j}}^{\mathsf{G}_n}(\mathbb{W}_{d_j})\right)^{\mathsf{G}_n} \cong \bigoplus_j \mathbb{W}_{d_j}^{\mathsf{G}_{d_j}},$$

Thus, $\dim \mathbb{F}_n^{\mathsf{G}_n}$ is constant for $n \geq k$. Moreover, by Proposition 2.3 and the fact that $\mathscr{K}$ and $\mathscr{U}$ are generated in degree k, we have $\dim \mathbb{K}_n^{\mathsf{G}_n} \geq \dim \mathbb{K}_{n+1}^{\mathsf{G}_{n+1}}$ and similarly $\dim \mathbb{U}_n^{\mathsf{G}_n} \geq \dim \mathbb{U}_{n+1}^{\mathsf{G}_{n+1}}$ for all $n \geq k$.

By the rank-nullity theorem, we have $\dim \mathbb{U}_n^{\mathsf{G}_n} = \dim \mathbb{F}_n^{\mathsf{G}_n} - \dim \mathbb{K}_n^{\mathsf{G}_n}$. As $\dim \mathbb{F}_n^{\mathsf{G}_n}$ is constant while both $\dim \mathbb{U}_n^{\mathsf{G}_n}$ and $\dim \mathbb{K}_n^{\mathsf{G}_n}$ are nonincreasing for $n \geq k$, we conclude that they are all constant for $n \geq k$. To conclude, we note that $\mathcal{P}_{\mathbb{U}_n}$ is injective when restricted to $\mathbb{U}_{n+1}^{\mathsf{G}_{n+1}}$ for all $n \geq k$ by Proposition 2.3. □

Remark 2.10 ($\mathscr{V}$-modules vs. centralizing subgroups)**.** *The definition of presentation degree assumes a "base" consistent sequence $\mathscr{V}$. Note however that it depends only on the centralizing subgroups $\{\mathsf{H}_{n,d}\}$ of*

$\mathscr{V}$. In fact, any sequence of subgroups $\{\mathsf{H}_{n,d} \subseteq \mathsf{G}_n\}_{d\leq n}$ satisfying $\mathsf{H}_{n+1,d} \supseteq \mathsf{H}_{n,d}$, $\mathsf{H}_{n,d+1} \subseteq \mathsf{H}_{n,d}$, and $\mathsf{H}_{n+1,d} \cap \mathsf{G}_n = \mathsf{H}_{n,d}$ for $d \leq n$ arise as centralizing subgroups of such a consistent sequence.

The centralizing subgroups play a central role because they determine embeddings $\{g\varphi_{n-1}\cdots\varphi_d\}_{g\in\mathsf{G}_n} \cong \mathsf{G}_n/\mathsf{H}_{n,d}$ of $\mathbb{V}_d$ into $\mathbb{V}_n$, and the combinatorics of these embeddings yields Theorem 2.11. See the proof of [49, Prop. 2.3.6] and Appendix A for example. We formulate our results in terms of $\mathscr{V}$-modules rather than the subgroups $\{\mathsf{H}_{n,d}\}$ directly because the sequences we use are often constructed from the same base sequence as in Section 2.3 below, easing the application of our results.

2.3 Constructions of Consistent Sequences

Expressive families of freely described convex sets require complex description spaces, and in turn complex consistent sequences. In this section, we describe common operations that yield complicated consistent sequences from simpler ones, along with a calculus for bounding the generation and presentation degrees of the resulting sequences.

Fix a family of group $\mathscr{G} = \{\mathsf{G}_n\}_{n\in\mathbb{N}}$ such that $\mathsf{G}_n \subseteq \mathsf{G}_{n+1}$. Suppose $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$ and $\mathscr{U} = \{(\mathbb{U}_n, \psi_n)\}$ are consistent sequences of $\mathscr{G}$-representations. Then the following are also consistent sequences of $\mathscr{G}$-representations:

(Sums) The *direct sum* of $\mathscr{V}$ and $\mathscr{U}$ is $\mathscr{V} \oplus \mathscr{U} = \{(\mathbb{V}_n \oplus \mathbb{U}_n, \varphi_n \oplus \psi_n)\}$.

If $\mathbb{W}$ is a fixed vector space, viewed as a trivial G_n-representation for all n, denote $\mathscr{V} \oplus \mathbb{W} = \{(\mathbb{V}_n \oplus \mathbb{W}, \varphi_n \oplus \mathrm{id}_{\mathbb{W}})\}$.

(Tensors) The *tensor product* of $\mathscr{V}$ and $\mathscr{U}$ is $\mathscr{V} \otimes \mathscr{U} = \{(\mathbb{V}_n \otimes \mathbb{U}_n, \varphi_n \otimes \psi_n)\}$.

This is also the sequence of spaces of *linear maps* $\mathcal{L}(\mathbb{V}_n, \mathbb{U}_n) \cong \mathbb{V}_n \otimes \mathbb{U}_n$, where we embed $A_n : \mathbb{V}_n \to \mathbb{U}_n$ to $(\varphi_n \otimes \psi_n)A_n = \psi_n A_n \varphi_n^* : \mathbb{V}_{n+1} \to \mathbb{U}_{n+1}$.

The *order-k tensors over $\mathscr{V}$* is $\mathscr{V}^{\otimes k}$.

If $\mathbb{W}$ is a fixed vector space, viewed as a trivial G_n-representation for all n, denote $\mathscr{V} \otimes \mathbb{W} = \{(\mathbb{V}_n \otimes \mathbb{W}, \varphi_n \otimes \mathrm{id}_{\mathbb{W}})\}$.

(Polynomials) The *degree-k polynomials over $\mathscr{V}$* is $\mathrm{Sym}^k\mathscr{V} = \{(\mathrm{Sym}^k\mathbb{V}_n, \varphi_n^{\otimes k})\}$, which is also the sequence of order-k symmetric tensors over $\mathscr{V}$. Here we view $\mathrm{Sym}^k\mathbb{V}_n \subseteq \mathbb{V}^{\otimes k}$ and restrict $\varphi_n^{\otimes k}$ to that subspace. The sequence of polynomials of degree at most k is denoted $\mathrm{Sym}^{\leq k}\mathscr{V} = \bigoplus_{j=1}^k \mathrm{Sym}^j\mathscr{V}$.

Similarly, we can form the sequence of kth exterior powers $\bigwedge^k \mathscr{V}$.

(Moments) The sequence of *moment matrices of order k over $\mathscr{V}$* is $\mathrm{Sym}^2(\mathrm{Sym}^{\leq k}\mathscr{V})$. Its elements can be viewed as symmetric matrices whose rows and columns are indexed by monomials of degree at most k in basis elements for $\mathscr{V}$.

(Images and Kernels) If $\{A_n \in \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n}\}$ is a morphism mapping $\mathscr{V} \to \mathscr{U}$, then the images $\mathrm{Im}\mathscr{A} = \{(A_n(\mathbb{V}_n), \psi_n)\}$ and kernels $\ker\mathscr{A} = \{(\ker A_n, \varphi_n)\}$ form consistent sequences.

If $\mathscr{V}, \mathscr{U}$ are $\mathscr{V}_0$-modules for some common consistent sequence $\mathscr{V}_0$, then all the above are $\mathscr{V}_0$-modules as well.

The group actions above are given in Section 1.3. The following theorem gives a calculus that allows us to bound the presentation degrees of sequences constructed from certain simpler ones with known presentation degrees. The following theorem is a consequence of results in [49, 50, 51] concerning calculus for generation degrees. We combine these results to obtain the following analogous calculus for presentation degrees, whose proof is given in Appendix A.

Theorem 2.11 (Calculus for generation and presentation degrees)**.** *Let $\mathscr{V}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations and let $\mathscr{W}, \mathscr{U}$ be $\mathscr{V}$-modules generated in degrees d_W, d_U and presented in degrees k_W, k_U, respectively. Then*

(Sums) *$\mathscr{W} \oplus \mathscr{U}$ is generated in degree* $\max\{d_W, d_U\}$ *and presented in degree* $\max\{k_W, k_U\}$.

(Images and kernels) *If $\mathscr{A} : \mathscr{W} \to \mathscr{U}$ and $\mathscr{A}^*$ are morphisms, then* $\mathrm{im}\mathscr{A}$ *and* $\ker\mathscr{A}$ *are generated in degree d_W and presented in degree k_W.*

If $\mathscr{V} = \{\mathbb{R}^n\}$ with $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, or S_n as in Example 1.5, we further have

(Tensors) $\mathscr{W} \otimes \mathscr{U}$ *is generated in degree* $d_W + d_U$ *and presented in degree* $\max\{k_W + d_U, k_U + d_W\}$.

(Sym and $\bigwedge$) $\mathrm{Sym}^\ell \mathscr{W}$, $\bigwedge^\ell \mathscr{W}$ *are generated in degree* ℓd_W *and presented in degree* $(\ell - 1)d_W + k_W$.

Proof. This follows from Theorem A.13 in the appendix. □

Example 2.12. *Suppose* $\{\mathbb{R}^n\}$ *as in Example 1.5 with* $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, *or* S_n. *Then* $(\mathbb{R}^n)^{\otimes k}$ *consists of* $n \times \cdots \times n$*-sized tensors with embeddings by zero padding and* $\mathrm{Sym}^k\mathbb{R}^n$ *consists of homogeneous polynomials of degree* k *in* n *variables. These are generated in degree* k *and presented in degree* k *if* $\mathsf{G}_n = \mathsf{B}_n, \mathsf{S}_n$ *and in degree* $k+1$ *if* $\mathsf{G}_n = \mathsf{D}_n$.

In this case, we also have $\mathrm{Sym}^2\mathbb{R}^n = \mathbb{S}^n$ *and* $\bigwedge^2 \mathbb{R}^n = \mathrm{Skew}(n)$. *The space* $\mathrm{Sym}^2(\mathrm{Sym}^{\leq k}\mathbb{R}^n)$ *can be viewed as symmetric matrices whose rows and columns are indexed by monomials of degree up to* k *in* n *variables. These sequences arises in optimization problems involving tensors, sums-of-squares polynomials, and moment sequences [64, 65].*

Remark 2.13 (Other groups). *Note that the last two conclusions in Theorem 2.11 fail without the restriction to* $\mathsf{G}_n = \mathrm{B}_n, \mathrm{D}_n$, *or* S_n. *For example, if we endow* $\mathscr{V} = \{\mathbb{R}^n\}$ *with the standard actions of the cyclic groups* $\mathsf{G}_n = \mathrm{Cyc}_n$ *and use embeddings by zero-padding, then* $\mathscr{V}$ *is generated in degree 1 but* $\mathscr{V}^{\otimes 2}$ *is not finitely-generated since* $\dim \mathcal{L}(\mathbb{R}^n)^{\mathrm{Cyc}_n} = n$ *does not stabilize.*

2.4 Permutation Modules

We introduce a class of particularly simple consistent sequences on which the group acts by permuting basis elements. These consistent sequences arise in our study of relative entropy cones and their constant-sized descriptions in Section 5.2. If a group G acts on a (finite) set $\mathcal{A}$, define $\mathbb{R}^{\mathcal{A}} = \bigoplus_{\alpha \in \mathcal{A}} e_\alpha$ to be the vector space with orthonormal basis elements $\{e_\alpha\}_{\alpha \in \mathcal{A}}$, which is a G-representation with action $g \cdot e_\alpha = e_{g \cdot \alpha}$.

Definition 2.14 (Permutation modules). *Let* $\mathscr{V} = \{\mathbb{V}_n\}$ *be a consistent sequence of* $\{\mathsf{G}_n\}$*-representations. Let* $\{\mathcal{A}_n \subseteq \mathbb{V}_n\}$ *be finite* G_n*-invariant sets satisfying* $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ *for all* n. *Then the* permutation $\mathscr{V}$-module *generated by the sets* $\{\mathcal{A}_n\}$ *is the* $\mathscr{V}$*-module* $\{\mathbb{R}^{\mathcal{A}_n}\}_n$.

Permutation modules can be analyzed in terms of the orbits in the sets $\mathcal{A}_n$. In particular, indicators of orbits form a basis for the space of invariants in a permutation module.

Proposition 2.15. *Let* $\mathscr{V} = \{\mathbb{V}_n\}$ *be a consistent sequence of* $\{\mathsf{G}_n\}$*-representations, let* $\{\mathcal{A}_n \subseteq \mathbb{V}_n\}$ *be a nested sequence of finite group-invariant sets, and let* $\mathscr{U} = \{\mathbb{U}_n = \mathbb{R}^{\mathcal{A}_n}\}$ *be the associated permutation* $\mathscr{V}$*-module.*

(a) $\mathscr{U}$ *is generated in degree* d *if and only if* $\mathcal{A}_n = \bigcup_{g \in \mathsf{G}_n} g\mathcal{A}_d$ *for all* $n \geq d$.

(b) The projections $\mathcal{P}_{\mathbb{U}_n} \colon \mathbb{U}_{n+1}^{\mathsf{G}_{n+1}} \to \mathbb{U}_n^{\mathsf{G}_n}$ *are isomorphisms for all* $n \geq d$ *if and only if (a) holds and the number of orbits in* $\mathcal{A}_n$, *which equals* $\dim \mathbb{U}_n^{\mathsf{G}_n}$, *is constant for all* $n \geq d$.

(c) Suppose $\mathscr{V} = \{\mathbb{R}^n\}$ *and* $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, *or* S_n *as in Example 1.5 and* $\mathscr{U}$ *is generated in degree* d. *Then* $\mathscr{U}$ *is free if* $\mathsf{G}_n = \mathsf{B}_n, \mathsf{S}_n$, *and agrees with a free module starting from degree* $d+1$ *if* $\mathsf{G}_n = \mathsf{D}_n$.

Here we say $\mathscr{U}$ agrees with a free module starting from degree d if there is a free $\mathscr{V}$-module $\mathscr{F} = \{\mathbb{F}_n\}$ and a morphism of sequences $\mathscr{F} \to \mathscr{U}$ such that $\mathbb{F}_n \to \mathbb{U}_n$ is an isomorphism for all $n \geq d$.

Proof. (a) We have $\mathbb{R}[\mathsf{G}_n]\mathbb{V}_d = \sum_{\substack{\alpha \in \mathcal{A}_d \\ g \in \mathsf{G}_n}} \mathbb{R}e_{g\alpha} = \bigoplus_{\alpha \in \bigcup_{g \in \mathsf{G}_n} g\mathcal{A}_d} \mathbb{R}e_\alpha$ which equals $\mathbb{V}_n = \bigoplus_{\alpha \in \mathcal{A}_n} \mathbb{R}e_\alpha$ precisely under the stated condition.

(b) If $\hat{\mathcal{A}}_n \subseteq \mathcal{A}_n$ is a set of G_n-orbit representatives, then $\mathbb{U}_n^{\mathsf{G}_n} \cong \mathbb{R}^{\hat{\mathcal{A}}_n}$ has a basis consisting of $\mathbb{1}_\alpha^{(n)} = \sum_{g \in \mathsf{G}_n / \mathsf{Stab}_{\mathsf{G}_n}(\alpha)} e_{g\alpha}$ for $\alpha \in \hat{\mathcal{A}}_n$, where $\mathsf{Stab}_{\mathsf{G}_n}(\alpha) = \{g \in \mathsf{G}_n : g \cdot \alpha = \alpha\}$. Moreover, we have $\mathcal{P}_{\mathbb{U}_n} \mathbb{1}_\alpha^{(n+1)} = \mathbb{1}_\alpha^{(n)}$ for each $\alpha \in \mathcal{A}_n$. Thus, $\mathcal{P}_{\mathbb{U}_n}$ is an isomorphism between invariants if and only if orbit representatives for $\mathcal{A}_n$ are also representatives for $\mathcal{A}_{n+1}$.

(c) Let $d_0 = 0$ if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ and $d_0 = 1$ if $\mathsf{G}_n = \mathsf{D}_n$. Let $\hat{\mathcal{A}} \subseteq \mathcal{A}_d$ be a set of minimal degree G_{d+d_0}-orbit representatives in $\mathcal{A}_{d+d_0}$. We argue that these are also the G_n-orbit representatives of $\mathcal{A}_n$ for all $n \geq d + d_0$. Indeed, since $\mathcal{A}_n = \bigcup_{g\in\mathsf{G}_n} g\mathcal{A}_d = \bigcup_{g\in\mathsf{G}_n} g\hat{\mathcal{A}}$, it suffices to show that distinct elements in $\hat{\mathcal{A}}$ lie in distinct G_n-orbits. To that end, observe that $\mathscr{V}$ satisfies

$$g \cdot \alpha \in \mathbb{V}_d \text{ for } \alpha \in \mathbb{V}_d,\ g \in \mathsf{G}_n \implies \exists \tilde{g} \in \mathsf{G}_{d+d_0} \text{ s.t. } g \cdot \alpha = \tilde{g} \cdot \alpha, \tag{3}$$

since if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ define $\tilde{g}$ to act as g on the coordinates $I = \{i \in [d] : \alpha_i \neq 0\}$ and act trivially on the others, and if $\mathsf{G}_n = \mathsf{D}_n$ and if g flips oddly-many signs of coordinates in I then in addition have $\tilde{g}$ flip the sign of coordinate $d+1$. Therefore, if $\alpha, \alpha' \in \mathcal{A}_d$ lie in the same G_n-orbit for $n > d + d_0$, then they also lie in the same G_{d+d_0}-orbit. Thus, we have $\mathbb{U}_n = \bigoplus_{\alpha\in\hat{\mathcal{A}}} \mathbb{R}[\mathsf{G}_n]e_\alpha$ for all $n \geq d + d_0$.

Next, we argue that $\mathscr{U}$ is free or agrees with a free module starting from degree $d + d_0$. Observe that $\mathscr{V}$ satisfies the additional property

$$\alpha \in \mathbb{V}_d \setminus \mathbb{V}_{d-1} \text{ has min. degree in its orbit } \implies \mathsf{Stab}_{\mathsf{G}_n}(\alpha) = \mathsf{Stab}_{\mathsf{G}_d}(\alpha)\mathsf{H}_{n,d}, \tag{4}$$

Indeed, if $\alpha \in \mathbb{V}_d \setminus \mathbb{V}_{d-1}$ has minimal degree then all its d entries are nonzero, hence any $g \in \mathsf{G}_n$ fixing α must fix the first d coordinates. Therefore, if $\alpha \in \mathbb{V}_d \setminus \mathbb{V}_{d-1}$ has minimal degree and $g_1, g_2, \ldots, g_M$ are coset representatives for $\mathsf{G}_n/\mathsf{Stab}_{\mathsf{G}_d}(\alpha)\mathsf{H}_{n,d}$, then

$$\mathbb{R}[\mathsf{G}_n]e_\alpha = \bigoplus_{m=1}^{M} \mathbb{R}e_{g_m\cdot\alpha} = \bigoplus_{m=1}^{M} g_m \cdot \mathbb{R}e_\alpha = \mathrm{Ind}_{\mathsf{Stab}_{\mathsf{G}_d}(\alpha)\mathsf{H}_{n,d}}^{\mathsf{G}_n}(\mathbb{R}e_\alpha) = \mathrm{Ind}_{\mathsf{G}_d\mathsf{H}_{n,d}}^{\mathsf{G}_n}\left(\mathrm{Ind}_{\mathsf{Stab}_{\mathsf{G}_d}(\alpha)}^{\mathsf{G}_d}\mathbb{R}e_\alpha\right),$$

where the first equality follows by (4), the second by the definition of a permutation representation (Definition 2.14), and the third equality follows by (Ind), and the last equality follows because $\mathsf{G}_d/\mathsf{Stab}_{\mathsf{G}_d}(\alpha) \cong (\mathsf{G}_d\mathsf{H}_{n,d})/(\mathsf{Stab}_{\mathsf{G}_d}(\alpha)\mathsf{H}_{n,d})$. Thus, if $\alpha \in \hat{\mathcal{A}}$ has degree d_α and we define $\mathbb{W}_\alpha = \mathrm{Ind}_{\mathsf{Stab}_{\mathsf{G}_{d_\alpha}}(\alpha)}^{\mathsf{G}_{d_\alpha}}\mathbb{R}e_\alpha$, then the map $\bigoplus_{\alpha\in\hat{\mathcal{A}}} \mathrm{Ind}_{\mathsf{G}_{d_\alpha}}(\mathbb{W}_\alpha)_n \to \mathbb{U}_n$ sending e_α to itself is an isomorphism for all $n \geq d + d_0$. Condition 3 shows that $\hat{\mathcal{A}} \cap \mathcal{A}_j$ is a set of minimal degree G_j-orbit representatives for each $j \leq d$ if $\mathsf{G}_n = \mathsf{B}_n, \mathsf{S}_n$, hence the above map is an isomorphism for all n. ☐

We remark that proposition 2.15(c) applies to any $\mathscr{V}$ satisfying (3) and (4).

2.5 Uniform Representation Stability

Proposition 2.3 shows that $\dim \mathbb{V}_n^{\mathsf{G}_n}$ stabilizes whenever $\{\mathbb{V}_n\}$ is a finitely-generated consistent sequence of $\{\mathsf{G}_n\}$-representations. In fact, the theory of [49, 50, 52] and others shows that for many standard families $\{\mathsf{G}_n\}$ of groups, the entire decomposition of $\mathbb{V}_n$ into irreducibles stabilizes. This phenomenon was called *uniform representation stability* in [16], and we use it to derive constant-sized descriptions for many sequences of PSD cones in Section 5.1. The following is a concrete instance of this phenomenon that we shall use there.

Theorem 2.16 ([49, Thm. 1.13],[50, Thm. 4.27]). *Let $\mathscr{V}_0 = \{\mathbb{R}^n\}$ with $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, or S_n be the consistent sequence from Example 1.5 and let $\mathscr{V} = \{\mathbb{V}_n\}$ be a $\mathscr{V}_0$-module generated in degree d and presented in degree k. Then there exists a finite set Λ and integers $(m_\lambda)_{\lambda\in\Lambda}$, together with an assignment $\lambda \mapsto \mathbb{W}_{\lambda[n]}$ of a distinct G_n-irreducible $\mathbb{W}_{\lambda[n]}$ to each $\lambda \in \Lambda$ for $n \geq k + d$ such that $\mathbb{V}_n \cong \bigoplus_{\lambda\in\Lambda} \mathbb{W}_{\lambda[n]}^{m_\lambda}$ as G_n-representations.*

Proof. The irreducibles of the groups $\mathsf{S}_n, \mathsf{D}_n, \mathsf{B}_n$ are indexed as in [50, §2.1], and the consistent labelling of irreducibles for different n is given in [50, §2.2]. Under this labelling, the $\mathscr{V}_0$-module $\mathscr{V}$ is uniformly representation stable with stable range $n \geq k + d$ by [50, Thms. 4.4, 4.27], which precisely says that the set of irreducibles appearing in the decomposition of the $\mathbb{V}_n$ and their multiplicities become constant for $n \geq k + d$ by [16, Def. 2.6]. ☐

Example 2.17. *Irreducibles of S_n are indexed by partitions of n. If $\lambda_1[n] = (n)$ is the trivial partition and $\lambda_2[n] = (n-1, 1)$, then $\mathbb{R}^n = \mathbb{W}_{\lambda_1[n]} \oplus \mathbb{W}_{\lambda_2[n]}$ for all $n \geq 1$, where $\mathbb{W}_{\lambda_1[n]} = \mathrm{span}\{\mathbb{1}_n\}$ and $\mathbb{W}_{\lambda_2[n]} = \{x \in \mathbb{R}^n : \mathbb{1}_n^\top x = 0\}$ are distinct irreducible representations of S_n.*

2.6 Stabilization of Shifted Sequences

Many of the phenomena in the representation stability literature, including Theorem 2.16, can be derived from properties of *shifted* consistent sequences, which are sequences with group actions restricted to centralizing subgroups, see [66] and references therein. In particular, we shall need the following result for such shifted sequences to derive constant-sized descriptions for relative entropy cones in Section 5.2.

Proposition 2.18 ([50, Lemma 4.19])**.** *Let $\mathscr{V} = \{\mathbb{R}^n\}$ and $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, or S_n as in Example 1.5 and let $\mathscr{U}$ be a $\mathscr{V}$-module presented in degree k. If $\{\mathsf{H}_{n,d}\}$ are the centralizing subgroups of $\mathscr{V}$ and $\ell \in \mathbb{N}$, then the projections $\mathcal{P}_{\mathbb{U}_n} \colon \mathbb{U}_{n+1}^{\mathsf{H}_{n+1,\ell}} \to \mathbb{U}_n^{\mathsf{H}_{n,\ell}}$ are isomorphisms for all $n \geq \ell + k$.*

Corollary 2.19. *Let $\mathscr{V} = \{\mathbb{R}^n\}$ and $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, or S_n as in Example 1.5, and let $\mathscr{U}$ be a $\mathscr{V}$-module presented in degree k. If $\beta \in \mathbb{R}^d \backslash \mathbb{R}^{d-1}$ has minimal degree in its G_d-orbit, then the projections $\mathcal{P}_{\mathbb{U}_n} \colon \mathbb{U}_{n+1}^{\mathsf{Stab}_{\mathsf{G}_{n+1}}(\beta)} \to \mathbb{U}_n^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)}$ are isomorphisms for all $n \geq d + k$.*

Proof. As shown in (4), we have $\mathsf{Stab}_{\mathsf{G}_n}(\beta) = \mathsf{Stab}_{\mathsf{G}_d}(\beta)\mathsf{H}_{n,d}$ for all $n \geq d$, hence

$$\mathbb{U}_n^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)} = \mathbb{U}_n^{\mathsf{H}_{n,d}} \cap \mathbb{U}_n^{\mathsf{Stab}_{\mathsf{G}_d}(\beta)}.$$

By Proposition 2.18, the projections $\mathcal{P}_{\mathbb{U}_n} \colon \mathbb{U}_{n+1}^{\mathsf{H}_{n+1,d}} \to \mathbb{U}_n^{\mathsf{H}_{n,d}}$ are isomorphisms for all $n \geq d + k$. It thus suffices to show that if $u \in \mathbb{U}_{n+1}^{\mathsf{H}_{n+1,d}}$ satisfies $\mathcal{P}_{\mathbb{U}_n} u \in \mathbb{U}_n^{\mathsf{Stab}_{\mathsf{G}_d}(\beta)}$, then $u \in \mathbb{U}_{n+1}^{\mathsf{Stab}_{\mathsf{G}_d}(\beta)}$. For any such u and $g \in \mathsf{Stab}_{\mathsf{G}_d}(\beta)$ we have $g \cdot u \in \mathbb{U}_{n+1}^{\mathsf{H}_{n+1,d}}$ because $\mathsf{H}_{n+1,d}$ and $\mathsf{Stab}_{\mathsf{G}_d}(\beta) \subseteq \mathsf{G}_d$ commute for our specific $\mathscr{V}$. As $\mathcal{P}_{\mathbb{U}_n}(u - g \cdot u) = 0$ and $\mathcal{P}_{\mathbb{U}_n}$ is injective on $\mathbb{U}_{n+1}^{\mathsf{H}_{n+1,d}}$, we get $u = g \cdot u$. □

2.7 Limits of Consistent Sequences

Lastly, we consider limits of consistent sequences. We shall use the following definitions and results to describe limits of freely-described sequences of convex sets in Section 3.3. We define limits of consistent sequences, and interpret our definitions in terms of these limits.

Definition 2.20. *For a consistent sequence $\mathscr{V} = \{\mathbb{V}_n\}$ of $\{\mathsf{G}_n\}$-representations, define its* limiting representation *as the vector space $\mathbb{V}_\infty = \bigcup_n \mathbb{V}_n$, viewed as a representation of $\mathsf{G}_\infty = \bigcup_n \mathsf{G}_n$.*

There is an approach to representation stability studying limiting representations of limiting groups as above, instead of representations of categories as in Appendix A. For example, the authors of [52] analyze representations of five standard infinite groups, including $\mathsf{O}_\infty, \mathsf{S}_\infty$, that occur as quotients or subrepresentations of tensor powers of $\mathbb{R}^\infty$ and its dual.

We remark that freely-described elements and morphisms of sequences can equivalently be defined in terms of limits. Indeed, given two consistent sequences $\{\mathbb{V}_n\}, \{\mathbb{U}_n\}$ of $\{\mathsf{G}_n\}$-representations, a sequence of equivariant linear maps $\{A_n \in \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n}\}$ is a morphism of sequences if and only if there exists $A_\infty \in \mathcal{L}(\mathbb{V}_\infty, \mathbb{U}_\infty)^{\mathsf{G}_\infty}$ satisfying $A_\infty|_{\mathbb{V}_n} = A_n$ for all n, i.e., iff $\{A_n\}$ extends to the limit. Similarly, a sequence of invariants $\{v_n \in \mathbb{V}_n^{\mathsf{G}_n}\}$ defines a sequence of invariant linear functionals $\ell_n(x) = \langle v_n, x\rangle \colon \mathbb{V}_n \to \mathbb{R}$, and these extend to a G_∞-invariant linear functional on $\mathbb{V}_\infty$ if and only if $\{v_n\}$ is a freely-described element. Every invariant functional on $\mathbb{V}_\infty$ arises in this way, so freely-described elements are in one-to-one correspondence with invariant linear functionals on the limit of a consistent sequence.

We also consider continuous limits of consistent sequences by taking the completion of $\mathbb{V}_\infty$ with respect to some norm. It is then natural to consider sequences of linear maps which extend to *continuous* maps between these limits. The inner products on each $\mathbb{V}_n$ extend to the limit $\mathbb{V}_\infty$, so we can always complete with respect to the induced norm and obtain a Hilbert space. For many of the examples we consider, however, we obtain more meaningful completions with respect to other norms. For the purpose of obtaining limiting descriptions of convex sets in this completion, we consider the following class of norms.

Definition 2.21 (Continuous limits)**.** *Let $\mathscr{V} = \{\mathbb{V}_n\}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations, let $\mathbb{V}_\infty = \bigcup_n \mathbb{V}_n$, and let $\mathcal{P}_n \colon \mathbb{V}_\infty \to \mathbb{V}_n$ be the orthogonal projection. Fix a norm $\|\cdot\|_c$ on $\mathbb{V}_\infty$ (not necessarily* *induced by the inner product) that satisfies* $\|\mathcal{P}_n x\|_c \leq C\|x\|_c$ *for some* $C > 0$ *and all* n *and* $x \in \mathbb{V}_\infty$*. We call the completion* $\overline{\mathbb{V}_\infty}$ *of* $\mathbb{V}_\infty$ *with respect to* $\|\cdot\|_c$ *a* continuous limit *of the sequence* $\mathscr{V}$*.*

Let $\{\mathbb{V}_n\}$ *and* $\{\mathbb{U}_n\}$ *be consistent sequences of* $\{\mathsf{G}_n\}$*-representations with associated continuous limits* $\overline{\mathbb{V}_\infty}$ *and* $\overline{\mathbb{U}_\infty}$*. A sequence of maps* $\{A_n : \mathbb{V}_n \to \mathbb{U}_n\}$ extends continuously to the limit *if there exists a bounded linear operator* $\overline{A_\infty} : \overline{\mathbb{V}_\infty} \to \overline{\mathbb{U}_\infty}$ *such that* $\overline{A_\infty}|_{\mathbb{V}_n} = A_n$ *for all* n*. A freely-described element* $\{u_n \in \mathbb{U}_n^{\mathsf{G}_n}\}$ extends continuously to the limit *if there exists* $u_\infty \in \overline{\mathbb{U}_\infty}$ *satisfying* $\mathcal{P}_n u_\infty = u_n$ *for all* n*.*

Note that a morphism of sequences $\{A_n : \mathbb{V}_n \to \mathbb{U}_n\}$ extends to the continuous limit if and only if the sequence of operator norms $\{\|A_n\|_{\mathrm{op},c}\}$ with respect to $\|\cdot\|_c$ is bounded.

Our condition on the norm $\|\cdot\|_c$ ensures that each projection $\mathcal{P}_n$ extends to a bounded linear map on the continuous limit, and that the $\mathcal{P}_n$ converge to the identity in the strong operator topology, as we show in the following lemma.

Lemma 2.22. *In the notation of Definition 2.21, there exists* $C > 0$ *such that* $\|\mathcal{P}_n x\|_c \leq C\|x\|_c$ *for all* n *and all* $x \in \overline{\mathbb{V}_\infty}$ *if and only if* $\mathcal{P}_n$ *extends to a bounded linear map* $\overline{\mathbb{V}_\infty} \to \mathbb{V}_n$ *for all* n *and* $\|x - \mathcal{P}_n x\|_c \to 0$ *as* $n \to \infty$ *for all* $x \in \overline{\mathbb{V}_\infty}$*.*

Proof. Suppose $\|\mathcal{P}_n x\|_c \leq C\|x\|_c$ for all n and x. Then $\|\mathcal{P}_n\|_{\mathrm{op},c} \leq C$ so that $\mathcal{P}_n$ extends to a bounded linear map on $\overline{\mathbb{V}_\infty}$ for all n. Furthermore, for each $x \in \overline{\mathbb{V}_\infty}$ there exists $x_n \in \mathbb{V}_n$ such that $\|x - x_n\|_c \to 0$. Since $x_n = \mathcal{P}_n x_n$, we have

$$\|x - \mathcal{P}_n x\|_c \leq \|x - x_n\|_c + \|\mathcal{P}_n x_n - \mathcal{P}_n x\|_c \leq (1 + C)\|x - x_n\|_c \to 0.$$

Conversely, if $\|\mathcal{P}_n\|_{\mathrm{op},c} < \infty$ for all n and $\mathcal{P}_n$ converge strongly to the identity, then for any $x \in \overline{\mathbb{V}_\infty}$ we have $\|\mathcal{P}_n x\|_c < \infty$ for all n and $\|\mathcal{P}_n x\|_c \to \|x\|_c$ as $n \to \infty$, hence $\sup_n \|\mathcal{P}_n x\|_c < \infty$. By the uniform boundedness principle, we have $\sup_n \|\mathcal{P}_n\|_{\mathrm{op},c} = C < \infty$. □

We close this section by noting that, more generally, if a sequence of convex functions $\{f_n : \mathbb{V}_n \to \mathbb{R} \cup \{\infty\}\}$ is intersection-compatible, then it extends to $f_\infty : \mathbb{V}_\infty \to \mathbb{R} \cup \{\infty\}$. By taking the closure of its epigraph it can further be extended to $\overline{f_\infty} : \overline{\mathbb{V}_\infty} \to \mathbb{R} \cup \{\infty\}$, in which case $\overline{\gamma_{C_\infty}} = \gamma_{\overline{C_\infty}}$ and $\overline{h_{C_\infty}} = h_{\overline{C_\infty}}$. Thus, the conic descriptions we obtain in Theorem 3.6 below for continuous limits of convex sets yield conic descriptions for limits of functions as well.

3 Structural Results on Free Descriptions

With the relevant background from representation stability in hand, we now proceed to prove the structural results discussed in Section 1.1.2 pertaining to freely-described convex sets. We leverage these results to provide a range of examples of free descriptions arising in various applications in Section 4. We begin with the following instructive example, which shows that freely-described sequences of convex sets need not satisfy either of our compatibility conditions. It also shows that a fixed convex set can extend to different freely-described sequences, depending on its conic description.

Example 3.1 ((In)compatible sequence of hypercubes)**.** *In this example, we construct two freely-described sequences of hypercubes with respect to the same description spaces, one of which is both intersection and projection compatible and the other is neither. Let* $\mathscr{V} = \{\mathbb{R}^n\}$ *with embeddings by zero-padding and the action of* $\mathsf{G}_n = \mathsf{B}_n$ *as in Example 1.5. Let* $s = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathsf{B}_n$*, which together with* S_n *generates* B_n*.*

We define two consistent sequences that will be used to construct our description spaces. Let $\mathscr{W}^{(1)} = \{\mathbb{R}^n\}$ *with embeddings by zero-padding but on which* s *acts trivially and* S_n *acts as usual. Let* $\mathscr{W}^{(2)} = \{\mathbb{R}^n \oplus \mathbb{R}^n\}$ *with embeddings by zero-padding each of the two summands, on which* B_n *acts as follows. We set* $s \cdot (x, y) = ([y_1, x_2, \ldots, x_n]^\top, [x_1, y_2, \ldots, y_n]^\top)$ *and* $\sigma \cdot (x, y) = (\sigma x, \sigma y)$ *for* $\sigma \in \mathsf{S}_n$*. We set our description spaces to*

$$\mathscr{U} = \mathscr{W}^{(1)} \oplus \mathscr{W}^{(2)} \oplus \mathbb{R} = \{(\mathbb{R}^n)^{\oplus 3} \oplus \mathbb{R}\}, \quad \mathscr{K} = \{0 \oplus (\mathbb{R}^n_+)^{\oplus 2} \oplus \mathbb{R}_+\}, \quad \mathscr{W} = \mathscr{W}^{(1)} \oplus \mathbb{R} = \{\mathbb{R}^n \oplus \mathbb{R}\}.$$

Consider the following two sequences of convex sets, both freely-described with respect to these description spaces. The first is the sequence of unit hypercubes $\{\mathcal{C}_n^{(1)} = [-1, 1]^n\}$ *given by* (ConicSeq) *with*

$$A_n^{(1)} x = (0, x, -x, 0), \qquad B_n^{(1)}(y, \beta) = (y, y, y, 0), \qquad u_n^{(1)} = (-\mathbb{1}_n, 0, 0, 0), \tag{5}$$

where $x, y \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. The second is the sequence of scaled hypercubes $\{\mathcal{C}_n^{(2)} = \frac{3}{2n-1}[-1,1]^n\}$ given by (ConicSeq) with

$$A_n^{(2)} = A_n^{(1)}, \qquad B_n^{(2)}(y,\beta) = (y - \mathbb{1}_n \mathbb{1}_n^\top y + \beta \mathbb{1}_n, y, y, -\mathbb{1}_n^\top y - \beta), \qquad u_n = (0,0,0,3). \tag{6}$$

It is straightforward to check that both sequences are freely-described. In particular, all the relevant spaces of invariants stabilize starting at $n = 2$, so that the extension of either description of $[-1,1]^2$ to a freely-described sequence is unique. However, the two sequences of sets have notably different properties. The first sequence is both intersection and projection compatible while the second is neither. Furthermore, note that $\mathcal{C}_2^{(1)} = \mathcal{C}_2^{(2)} = [-1,1]^2$ but $\mathcal{C}_n^{(1)} \neq \mathcal{C}_n^{(2)}$ for all $n \neq 2$. Thus, the extension of the unit square $[-1,1]^2$ to a freely-described sequence is not unique, but depends on its conic description.

Motivated by this example, we give conditions on free descriptions ensuring that the corresponding sequence of sets satisfies compatibility conditions. In turn, these yield conditions on a description of a fixed set ensuring that it extends to a freely-described and compatible sequence of sets as well as conditions on the existence of limiting descriptions.

3.1 Free Descriptions Certifying Compatibility

Example 3.1 shows that a freely-described sequence of convex sets need not satisfy either compatibility condition in Definition 1.8. In this section, we therefore give conditions on free descriptions ensuring that our compatibility conditions are satisfied.

Proposition 3.2. *Let $\{\mathbb{V}_n\}, \{\mathbb{W}_n\}, \{\mathbb{U}_n\}$ be consistent sequences of $\{\mathsf{G}_n\}$-representations, let $\{\mathcal{K}_n \subseteq \mathbb{U}_n\}$ be an intersection- and projection-compatible sequence of convex cones, and let $\mathscr{C} = \{\mathcal{C}_n \subseteq \mathbb{V}_n\}$ be described by linear maps $\{A_n : \mathbb{V}_n \to \mathbb{U}_n\}, \{B_n : \mathbb{W}_n \to \mathbb{U}_n\}$ and elements $\{u_n \in \mathbb{U}_n^{\mathsf{G}_n}\}$ as in (ConicSeq).*

(a) If $\{A_n\}, \{B_n\}, \{B_n^*\}$ are morphisms, $\{u_n\}$ is freely-described, and $u_{n+1} - u_n \in \mathcal{K}_{n+1}$ for all n, then $\mathscr{C}$ is intersection-compatible. If, in addition, $\{A_n^*\}$ is a morphism, then $\mathscr{C}$ is also projection-compatible.

(b) If $\{A_n\}, \{A_n^*\}\{B_n\}, \{B_n^*\}$ are morphisms, $\{u_n\}$ is freely-described, and $u_{n+1} - u_n \in \mathcal{K}_{n+1} + A_{n+1}(\mathbb{V}_n^\perp) + B_{n+1}(\mathbb{W}_n)$, then $\mathscr{C}$ is projection-compatible.

Proof. (a) We show $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$. If $x \in \mathcal{C}_n$ then there is $y \in \mathbb{W}_n$ satisfying $A_n x + B_n y + u_n \in \mathcal{K}_n$. Then

$$A_{n+1}x + B_{n+1}y + u_{n+1} = A_n x + B_n y + u_n + (u_{n+1} - u_n) \in \mathcal{K}_{n+1},$$

where we used the facts that $\{A_n\}, \{B_n\}$ are morphisms, that $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$ by intersection-compatiblity, and that $u_{n+1} - u_n \in \mathcal{K}_{n+1}$. Thus, $x \in \mathcal{C}_{n+1}$. Next, we show $\mathcal{C}_{n+1} \cap \mathbb{V}_n \subseteq \mathcal{C}_n$. If $x \in \mathcal{C}_{n+1} \cap \mathbb{V}_n$, there exists $y \in \mathbb{W}_{n+1}$ satisfying $A_{n+1}x + B_{n+1}y + u_{n+1} \in \mathcal{K}_{n+1}$. Because $\{A_n\}$ is a morphism, we have $A_{n+1}x = A_n x$ and hence $\mathcal{P}_{\mathbb{U}_n} A_{n+1} x = A_n x$. Because $\{B_n^*\}$ is a morphism, we have $\mathcal{P}_{\mathbb{U}_n} B_{n+1} = B_n \mathcal{P}_{\mathbb{W}_n}$, hence $\mathcal{P}_{\mathbb{U}_n} B_{n+1} y = B_{n+1}(\mathcal{P}_{\mathbb{W}_n} y)$. Finally, we have $\mathcal{P}_{\mathbb{U}_n} u_{n+1} = u_n$ because $\{u_n\}$ is freely-described, and $\mathcal{P}_{\mathbb{U}_n} \mathcal{K}_{n+1} \subseteq \mathcal{K}_n$ because $\{\mathcal{K}_n\}$ is projection-compatible. Thus, applying $\mathcal{P}_{\mathbb{U}_n}$ we obtain $A_n x + B_n(\mathcal{P}_{\mathbb{W}_n} y) + u_n \in \mathcal{K}_n$, thus yielding that $x \in \mathcal{C}_n$. We conclude that $\mathscr{C}$ is intersection-compatible.

Because $\mathscr{C}$ is intersection-compatible, we have $\mathcal{P}_{\mathbb{V}_n} \mathcal{C}_{n+1} \supseteq \mathcal{C}_n$. Conversely, if $x \in \mathcal{C}_{n+1}$ then $A_{n+1}x + B_{n+1}y + u_{n+1} \in \mathcal{K}_{n+1}$ for some $y \in \mathbb{W}_{n+1}$. If $\{A_n^*\}$ is a morphism, then $\mathcal{P}_{\mathbb{U}_n} A_{n+1} = A_n \mathcal{P}_{\mathbb{V}_n}$. Applying $\mathcal{P}_{\mathbb{U}_n}$ to both sides we obtain $A_n \mathcal{P}_{\mathbb{V}_n} x + B_n \mathcal{P}_{\mathbb{W}_n} y + u_n \in \mathcal{K}_n$ and hence $\mathcal{P}_{\mathbb{V}_n} x \in \mathcal{C}_n$, which implies that $\mathscr{C}$ is projection-compatible.

(b) First, we show $\mathcal{P}_{\mathbb{V}_n} \mathcal{C}_{n+1} \subseteq \mathcal{C}_n$. If $x \in \mathcal{C}_{n+1}$ then there is $y \in \mathbb{W}_{n+1}$ satisfying $A_{n+1}x + B_{n+1}y + u_{n+1} \in \mathcal{K}_{n+1}$. Applying $\mathcal{P}_{\mathbb{U}_n}$ to both sides and using the facts that $\{A_n^*\}, \{B_n^*\}$ are morphisms, that $\{u_n\}$ is freely-described, and that $\{\mathcal{K}_n\}$ is projection-compatible, we obtain $A_n(\mathcal{P}_{\mathbb{V}_n} x) + B_n(\mathcal{P}_{\mathbb{W}_n} y) + u_n \in \mathcal{K}_n$, showing that $\mathcal{P}_{\mathbb{V}_n} x \in \mathcal{C}_n$. Second, we show $\mathcal{C}_n \subseteq \mathcal{P}_{\mathbb{V}_n} \mathcal{C}_{n+1}$. Suppose $A_n x + B_n y + u_n \in \mathcal{K}_n$ for $x \in \mathbb{V}_n$. Let $x_\perp \in \mathbb{V}_n^\perp$ and $y' \in \mathbb{W}_n$ satisfy $u_{n+1} - u_n + A_{n+1}x_\perp + B_{n+1}y' \in \mathcal{K}_{n+1}$. As $\{A_n\}, \{B_n\}$ are morphisms,

$$A_{n+1}(x + x_\perp) + B_{n+1}(y + y') + u_{n+1} = A_n x + B_n y + u_n + (u_{n+1} - u_n + A_{n+1}x_\perp + B_{n+1}y') \in \mathcal{K}_{n+1},$$

hence $x + x_\perp \in \mathcal{C}_{n+1}$ and $\mathcal{P}_{\mathbb{V}_n}(x + x_\perp) = x$. This shows $\mathscr{C}$ is projection-compatible. $\square$

We say that a sequence of conic descriptions *certifies compatibility* when it satisfies the hypotheses of Proposition 3.2.

Remark 3.3. *We make a number of remarks about the conditions in Proposition 3.2.*

- *Standard sequences of cones such as nonnegative orthants and PSD cones satisfy both intersection and projection compatibility.*
- *The set of linear maps* $\{A_n\}$ *and* $\{B_n\}$ *satisfying the hypotheses of Proposition 3.2(a) form* linear subspaces *of the corresponding spaces of freely-described elements. The set of sequences* $\{u_n\}$ *satisfying those hypotheses form a* convex cone. *Similarly, we can parametrize descriptions satisfying the hypotheses of Proposition 3.2(b) by a convex cone, by considering* $\{u_n\}$ *of the form* $u_n = A_n(v_n)+B_n(w_n)+z_n$ *where* $\{v_n \in \mathbb{V}_n^{\mathsf{G}_n}\}$, $\{w_n \in \mathbb{W}_n^{\mathsf{G}_n}\}$, *and* $\{z_n \in \mathcal{K}_n^{\mathsf{G}_n}\}$ *are freely-described.*
- *Freely-described sets satisfying the hypotheses of Proposition 3.2(b) need not be intersection-compatible, as the elliptope in (17) studied below demonstrates.*
- *Free descriptions certifying compatibility often extend to descriptions of infinite-dimensional limits, see Theorem 3.6 below.*

Returning to Example 3.1, we note that the description (5) satisfies all the hypotheses of Proposition 3.2(a) and hence certifies the intersection and projection compatibility of the sequence of hypercubes it describes. On the other hand, the description (6) does not satisfy the hypotheses of either part of the above theorem because neither $\{B_n^{(2)}\}$ nor $\{(B_n^{(2)})^*\}$ are morphisms.

3.2 Extending a Convex Set to a Freely-Described Sequence

Let $\mathscr{V} = \{\mathbb{V}_n\}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations. In this section, we fix n_0 and seek to extend a convex subset $\mathcal{C}_{n_0} \subseteq \mathbb{V}_{n_0}$ to a freely-described and compatible sequence. As we saw in Example 3.1, different extensions may be obtained from different fixed-dimensional conic descriptions of $\mathcal{C}_{n_0}$. We therefore further fix consistent sequences $\mathscr{U} = \{\mathbb{U}_n\}, \mathscr{W} = \{\mathbb{W}_n\}$ and cones $\{\mathcal{K}_n \subseteq \mathbb{U}_n\}$ satisfying both intersection and projection compatibility, and assume $\mathcal{C}_{n_0}$ is described by (ConicSeq). We now ask when the description of $\mathcal{C}_{n_0}$ can be extended to a free description that yields a compatible sequence of convex sets.

If n_0 exceeds the presentation degrees of $\mathscr{W}, \mathscr{U}, \mathscr{V}\otimes\mathscr{U}$ and $\mathscr{W}\otimes\mathscr{U}$, then Proposition 2.9 shows that we can uniquely extend all the elements in the description of $\mathcal{C}_{n_0}$ to freely-described elements, and hence extend $\mathcal{C}_{n_0}$ to a freely-described sequence. To ensure that this unique extension satisfies our compatibility conditions, we give conditions on these invariants guaranteeing that their extensions to freely-described elements satisfy the conditions of Proposition 3.2.

We begin by characterizing when a fixed equivariant map $A_{n_0} \in \mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ extends to a morphism of sequences $\{A_n \in \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n}\}$.

Theorem 3.4. *Let* $\mathscr{V}_0$ *be a consistent sequence of* $\{\mathsf{G}_n\}$*-representations and let* $\mathscr{V} = \{\mathbb{V}_n\}, \mathscr{U} = \{\mathbb{U}_n\}$ *be* $\mathscr{V}_0$*-modules. Assume* $\mathscr{V}$ *is generated in degree* d *and presented in degree* k*, and fix* $n_0 \geq k$*. Then* A_{n_0} *extends to a morphism of sequences if and only if* $A_{n_0} \in \mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ *satisfies* $A_{n_0}(\mathbb{V}_j) \subseteq \mathbb{U}_j$ *for* $j \leq d$.

Proof. If A_{n_0} extends to a morphism then $A_{n_0}(\mathbb{V}_j) = A_j(\mathbb{V}_j) \subseteq \mathbb{U}_j$ for all $j \leq n_0$. Conversely, assume $A_{n_0}(\mathbb{V}_j) \subseteq \mathbb{U}_j$ for $j \leq d$, and let $\{\mathsf{H}_{n,d}\}$ be the centralizing subgroups of $\mathscr{V}_0$. Suppose first that $\mathscr{V} = \mathscr{F} = \bigoplus_j \mathrm{Ind}_{\mathsf{G}_{d_j}} \mathbb{W}_{d_j}$ is free. Note that it is generated in degree $\max_j d_j \leq d$.

Let $A_{d_j} = A_{n_0}|_{\mathbb{W}_{d_j}}$ and fix $n \geq d_j$. Because $\mathscr{U}$ is a $\mathscr{V}_0$-module, we have $\mathbb{U}_{d_j} \subseteq \mathbb{U}_n^{\mathsf{H}_{n,d_j}}$, so we can view $\mathbb{U}_{d_j}$ as a representation of $\mathsf{G}_{d_j}\mathsf{H}_{n,d_j}$ on which H_{n,d_j} acts trivially. As $A_{d_j}(\mathbb{W}_{d_j}) \subseteq \mathbb{U}_{d_j}$ and is $\mathsf{G}_{d_j}\mathsf{H}_{n,d_j}$-equivariant, the following composition defines an equivariant map

$$A_{n,j} \colon \mathrm{Ind}_{\mathsf{G}_{d_j}\mathsf{H}_{n,d_j}}^{\mathsf{G}_n}(\mathbb{W}_{d_j}) \xrightarrow{\mathrm{Ind}(A_{d_j})} \mathrm{Ind}_{\mathsf{G}_{d_j}\mathsf{H}_{n,d_j}}^{\mathsf{G}_n} \mathbb{U}_{d_j} \xrightarrow{g\otimes u \mapsto g\cdot u} \mathbb{U}_n,$$

where the induced map $\mathrm{Ind}(A_{d_j})$ was defined in Section 1.3. Note that $A_{n_0,j} = A_{n_0}|_{\mathrm{Ind}_{\mathsf{G}_{d_j}}(\mathbb{W}_{d_j})_{n_0}}$, since $A_{n_0,j}(g\otimes w) = g\cdot A_{n_0}w$ for all $g \in \mathsf{G}_n$ and $w \in \mathbb{W}_{d_j}$. Also, $\{A_{n,j}\}$ defines a morphism $\mathrm{Ind}_{\mathsf{G}_{d_j}}(\mathbb{W}_{d_j}) \to \mathscr{U}$. Therefore, the desired extension of A_{n_0} to a morphism $\{A_n\}$ is given by $A_n = \bigoplus_j A_{n,j} \colon \mathbb{V}_n \to \mathbb{U}_n$.

Now suppose $\mathscr{F} = \{\mathbb{F}_n\}$ is an algebraically free $\mathscr{V}$-module as above with a surjection $\mathscr{F} \to \mathscr{V}$ whose kernel $\mathscr{K} = \{\mathbb{K}_n\}$ is generated in degree k. Define the composition

$$\widetilde{A}_{n_0} \colon \mathbb{F}_{n_0} \to \mathbb{V}_{n_0} \xrightarrow{A_{n_0}} \mathbb{U}_{n_0},$$

which satisfies $\widetilde{A}_{n_0}(\mathbb{F}_j) \subseteq \mathbb{U}_j$ for all $j \leq d$ by assumption and $\widetilde{A}_{n_0}(\mathbb{K}_{n_0}) = 0$ by its definition. By the previous paragraph, it extends to a morphism $\{\widetilde{A}_n \colon \mathbb{F}_n \to \mathbb{U}_n\}$. Because $\mathscr{K}$ is generated in degree k and $n_0 \geq k$, we have $\mathbb{K}_n = \mathbb{R}[\mathsf{G}_n]\mathbb{K}_{n_0}$. Because $\widetilde{A}_n$ is equivariant, we have $\widetilde{A}_n(\mathbb{K}_n) = 0$. Therefore, $\widetilde{A}_n$ can be factored as $\mathbb{F}_n \to \mathbb{F}_n/\mathbb{K}_n = \mathbb{V}_n \xrightarrow{A_n} \mathbb{U}_n$, where the maps A_n in this factorization give the desired extension of A_{n_0} to a morphism $\mathscr{V} \to \mathscr{U}$. □

To satisfy the conditions in Proposition 3.2, we also use Theorem 3.4 to ensure $\{A_n^*\}$ defines a morphism. To that end, note that $A_{n_0}^*(\mathbb{U}_j) \subseteq \mathbb{V}_j$ if and only if $A_{n_0}(\mathbb{V}_j^\perp) \subseteq \mathbb{U}_j^\perp$, where orthogonal complements are taken inside $\mathbb{V}_{n_0}$ and $\mathbb{U}_{n_0}$. We can now give conditions guaranteeing extendability of a convex set to a freely-described and compatible sequence.

Theorem 3.5 (Parametrizing freely-described and compatible sequences)**.** *Let $\mathscr{V}_0$ be a consistent sequence of $\{\mathsf{G}_n\}$ representations and let $\mathscr{V} = \{\mathbb{V}_n\}$, $\mathscr{W} = \{\mathbb{W}_n\}$, and $\mathscr{U} = \{\mathbb{U}_n\}$ be $\mathscr{V}_0$-modules generated in degrees d_V, d_U, d_W, respectively, and presented in degree k. Let $\{\mathcal{K}_n \subseteq \mathbb{U}_n\}$ be an intersection and projection-compatible sequence of convex cones. Fix $n_0 \geq k$.*

Let $u_{n_0} \in \mathbb{U}_{n_0}^{\mathsf{G}_{n_0}}$ and let $A_{n_0} \in \mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$, $B_{n_0} \in \mathcal{L}(\mathbb{W}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ such that

$$A_{n_0}(\mathbb{V}_j) \subseteq \mathbb{U}_j \text{ for } j \leq d_V, \quad B_{n_0}(\mathbb{W}_j) \subseteq \mathbb{U}_j \text{ for } j \leq d_W, \quad B_{n_0}(\mathbb{W}_j^\perp) \subseteq \mathbb{U}_j^\perp \text{ for } j \leq d_U. \tag{7}$$

Then there are unique extensions of A_{n_0} and B_{n_0} to morphisms $\{A_n \in \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n}\}$ and $\{B_n \in \mathcal{L}(\mathbb{W}_n, \mathbb{U}_n)^{\mathsf{G}_n}\}$, where $\{B_n^*\}$ is a morphism as well. Furthermore, there is a unique extension of u_{n_0} to a freely-described element $\{u_n \in \mathbb{U}_n^{\mathsf{G}_n}\}$. Let $\mathscr{C} = \{\mathcal{C}_n\}$ be the freely-described sequence of convex sets given by (ConicSeq).

(a) If $u_{n+1} - u_n \in \mathcal{K}_{n+1}$ for all n, then $\mathscr{C}$ is intersection-compatible. If, in addition, we have $A_{n_0}(\mathbb{V}_j^\perp) \subseteq \mathbb{U}_j^\perp$ for $j \leq d_U$, then $\mathscr{C}$ is also projection-compatible.

(b) If $u_{n+1} - u_n \in \mathcal{K}_{n+1} + A_{n+1}(\mathbb{V}_n^\perp) + B_{n+1}(\mathbb{U}_{n+1})$ for all n, then $\mathscr{C}$ is projection-compatible.

Proof. Theorem 3.4 shows that A_{n_0} and B_{n_0} uniquely extend to morphisms $\{A_n\}, \{B_n\}$ such that $\{B_n^*\}$ is also a morphism. Proposition 2.9 shows that u_{n_0} extends to a freely-described element $\{u_n\}$. Under the stated conditions on these extensions, Proposition 3.2 yields the claimed compatibility conditions for the sequence of convex sets $\{\mathcal{C}_n\}$ given by (ConicSeq). □

In Section 6, we use Theorem 3.5 to computationally parametrize and search over free descriptions certifying compatibility.

3.3 Limits of Freely-Described Convex Sets

Our last structural result gives conditions under which free descriptions extend to descriptions of continuous limits of convex sets. The certificates of compatibility in Proposition 3.2 play a major role once again in the existence of these limiting descriptions. If $\mathscr{C} = \{\mathcal{C}_n \subseteq \mathbb{V}_n\}$ is an intersection-compatible sequence of convex subsets of a consistent sequence $\{\mathbb{V}_n\}$, define $\mathcal{C}_\infty = \bigcup_n \mathcal{C}_n$, which is a convex subset of $\mathbb{V}_\infty$.

Theorem 3.6 (Descriptions of limits)**.** *Suppose $\mathscr{C} = \{\mathcal{C}_n \subseteq \mathbb{V}_n\}$ is given by (ConicSeq). If all the hypotheses of Proposition 3.2(a) are satisfied (so $\mathscr{C}$ is intersection and projection compatible) and if $\{A_n\}, \{B_n\}, \{u_n\}$ extend continuously to the limit, then*

$$\left\{x \in \overline{\mathbb{V}_\infty} : \exists y \in \overline{\mathbb{W}_\infty} \text{ s.t. } \overline{A_\infty}x + \overline{B_\infty}y + u_\infty \in \overline{\mathcal{K}_\infty}\right\}, \tag{8}$$

contains $\mathcal{C}_\infty$ and is dense in its closure. If $B_\infty = 0$, then (8) equals $\overline{\mathcal{C}_\infty}$.

Proof. To prove that $\mathcal{C}_\infty$ is contained in (8), observe that $u_\infty - u_n = \lim_{N\to\infty}(u_N - u_n) \in \overline{\mathcal{K}_\infty}$ for all n by Lemma 2.22. Therefore, if $x \in \mathcal{C}_n$ and $y \in \mathbb{W}_n$ satisfy $A_n x + B_n y + u_n \in \mathcal{K}_n$, then $\overline{A_\infty} x + \overline{B_\infty} y + u_\infty = A_n x + B_n y + u_n + (u_\infty - u_n) \in \overline{\mathcal{K}_\infty}$, proving that x is in (8).

To prove that (8) is contained in $\overline{\mathcal{C}_\infty}$, suppose $x \in \overline{\mathbb{V}_\infty}$ and $y \in \overline{\mathbb{W}_\infty}$ satisfy $\overline{A_\infty} x + \overline{B_\infty} y + u_\infty \in \overline{\mathcal{K}_\infty}$. Because $\{A_n^*\}, \{B_n^*\}$ are morphisms and $\{\mathcal{K}_n\}$ is projection-compatible, applying $\mathcal{P}_{\mathbb{U}_n}$ we obtain $A_n(\mathcal{P}_{\mathbb{V}_n} x) + B_n(\mathcal{P}_{\mathbb{W}_n} y) + u_n \in \mathcal{K}_n$, hence $\mathcal{P}_{\mathbb{V}_n} x \in \mathcal{C}_n$ for all n and $x = \lim_n \mathcal{P}_{\mathbb{V}_n} x \in \overline{\mathcal{C}_\infty}$ by Lemma 2.22.

If $B_\infty = 0$ then (8) is the preimage under the continuous map $x \mapsto \overline{A_\infty} x + u_\infty$ of the closed cone $\overline{\mathcal{K}_\infty}$, hence (8) is closed and must equal $\overline{\mathcal{C}_\infty}$. ☐

If the set (8) is dense in $\overline{\mathcal{C}_\infty}$, then optimizing a continuous function over (8) and over $\overline{\mathcal{C}_\infty}$ are equivalent, yielding an finitely-parametrized conic program in $\overline{\mathbb{V}_\infty}$.

4 Examples

In this section, we present examples of freely-described and compatible sequences of sets and functions arising in a variety of applications. For several of these, we derive finitely-parametrized families of freely-described sets, and we also give limiting descriptions and some related consequences.

4.1 Simplices and Norms

Let $\mathscr{V} = \{\mathbb{V}_n = \mathbb{R}^n\}$ with embedding by zero-padding and the action of $\mathsf{G}_n = \mathsf{S}_n$ as in Example 1.5.

Simplex: The sequence of simplices $\Delta^{n-1} = \{x \in \mathbb{R}^n : x \geq 0,\ \mathbb{1}_n^\top x = 1\}$ is freely-described, as the associated description is of the form (ConicSeq) with description spaces $\mathscr{U} = \mathscr{V} \oplus \mathbb{R} = \{\mathbb{U}_n = \mathbb{R}^{n+1}\}$, $\mathscr{W} = \{\mathbb{W}_n = 0\}$ and cones $\mathscr{K} = \{\mathcal{K}_n = \mathbb{R}^n_+ \oplus \{0\}\}$, and with $A_n = [I_n, -\mathbb{1}_n]^\top$, $B_n = 0$, and $u_n = [0,1]^\top$. Moreover, the simplices $\{\Delta^{n-1}\}$ are intersection-compatible (but not projection-compatible); the above description satisfies the hypotheses of Proposition 3.2(a) and hence certifies this compatibility.

Solid simplex: In contrast, consider the sequence of solid simplices $\Delta^n_s = \{x \in \mathbb{R}^n : x \geq 0, \mathbb{1}_n^\top x \leq 1\}$. This sequence of conic descriptions is the same as above except that here $\mathcal{K}_n = \mathbb{R}^{n+1}_+$, and in particular is free and certifies intersection compatibility of $\{\Delta^n_s\}$. However, the sequence $\{\Delta^n_s\}$ is also projection-compatible, but the above free description of it does *not* certify this compatibility. That is because $\{A_n^*\}$ is not a morphism, hence the hypotheses of the second part of Proposition 3.2(a) are not satisfied. Indeed, the above description of the solid simplex does not make its projection compatibility apparent, since $\mathbb{1}_n^\top \mathcal{P}_{\mathbb{V}_n} x$ can be arbitrary compared to $\mathbb{1}_{n+1}^\top x$ for general $x \in \mathbb{R}^{n+1}$, but must be smaller if $x \geq 0$.

Instead, the following is a free *semidefinite* description of the solid simplices that certifies both intersection and projection compatibilities:

$$\Delta^n_s = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} 1 & x^\top \\ x & \mathrm{diag}(x) \end{bmatrix} = A_n x + u_n \succeq 0 \right\}, \tag{9}$$

which is of the form (ConicSeq) with $\mathscr{U} = \mathrm{Sym}^2(\mathrm{Sym}^{\leq 1}\mathscr{V}) = \{\mathbb{S}^{n+1}\}$, $\mathscr{K} = \{\mathbb{S}^{n+1}_+\}$, and $\mathscr{W} = 0$. Here both $\{A_n\}$ and $\{A_n^*\}$ are morphisms. It would be interesting to find a free linear programming description of the solid simplices which certifies both compatibilities, or to show that none exists.

Norm balls: Several related free descriptions are derived from the above. The sequence of ℓ_1 unit balls $\mathcal{B}^n_{\ell_1} = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$ is intersection and projection compatible. It can be written as $\mathcal{B}^n_{\ell_1} = \{x \in \mathbb{R}^n : \exists y \in \Delta^n_s \text{ s.t. } -y \leq x \leq y\}$, which combined with (9) yields a free description certifying both compatibilities. Similarly, the sequence of ℓ_2 unit balls $\mathcal{B}^n_{\ell_2} = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ is intersection and projection compatible, and its standard SDP description $\mathcal{B}^n_{\ell_2} = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} 1 & x^\top \\ x & I_n \end{bmatrix} = A_n x + u_n \succeq 0 \right\}$ is free and certifies these compatibilities similarly to (9).

Notice that both sequences of norm balls are invariant under the larger group $\mathsf{G}_n = \mathsf{B}_n$. All of the above descriptions are in fact equivariant under this larger group, when $\mathscr{U}, \mathscr{W}$ are endowed with the corresponding group action.

4.2 Regular Polygons

The following example illustrates a natural sequence of convex sets that is freely-described but satisfies neither intersection nor projection compatibility. Let $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$ be the consistent sequence $\mathbb{V}_n = \mathbb{R}^2$ with $\varphi_n = \mathrm{id}_{\mathbb{R}^2}$ and the standard action of the dihedral group $\mathsf{G}_n = \mathrm{Dih}_{2^n}$. Consider the sequence of regular 2^n-gons $\mathscr{C} = \{\mathcal{C}_n \subseteq \mathbb{R}^2\}$ defined by

$$\mathcal{C}_n = \mathrm{conv}\left\{\begin{bmatrix}\cos\theta_i\\ \sin\theta_i\end{bmatrix}\right\},\quad \theta_i = \frac{2\pi i}{2^n},\quad i \in \{0,\dots,2^n-1\}. \tag{10}$$

Because $\mathbb{V}_n = \mathbb{V}_{n+1}$ while $\mathcal{C}_n \neq \mathcal{C}_{n+1}$, the sequence $\mathscr{C}$ satisfies neither intersection nor projection compatibility. Nevertheless, it admits the free description

$$\mathcal{C}_n = \left\{ x \in \mathbb{R}^2 : \exists y \in \mathbb{R}^{2^n} \text{ s.t. } \begin{bmatrix}-I\\ 0\\ 0\end{bmatrix} x + \begin{bmatrix}\begin{bmatrix}\cdots & \cos(2\pi i/2^n) & \cdots\\ & \sin(2\pi i/2^n) & \end{bmatrix}\\ I_{2^n}\\ \mathbb{1}_{2^n}^\top\end{bmatrix} y + \begin{bmatrix}0\\ 0\\ -1\end{bmatrix} \in 0 \oplus \mathbb{R}^n_+ \oplus 0 \right\},$$

where $\mathscr{W} = \{\mathbb{R}^{2^n}\}_n$ with embeddings $y \mapsto y \otimes [1,0]^\top$, and $\mathscr{U} = \mathscr{V} \oplus \mathscr{W} \oplus \mathbb{R}$. We put the standard inner products on $\mathbb{R}^{2^n}$. The group permutes the 2^n vertices of $\mathcal{C}_n$, defining a permutation action on $[2^n]$, and it acts on $\mathbb{R}^{2^n}$ by applying these permutations to coordinates.

If $\theta_i = \pi(2i+1)/2^n$ in (10) instead, we mention that the semidefinite description of $\mathcal{C}_n$ given in [40] is also free when $\mathscr{U}, \mathscr{W}, \mathscr{K}$ are chosen appropriately.

4.3 Permutahedra, Schur-Horn orbitopes, and Limits

Permutahedra: Consider the sequence of standard permutahedra

$$\mathrm{Perm}_n = \mathrm{conv}\left\{g \cdot [1,2,\dots,n]^\top\right\} = \left\{M[1,2,\dots,n]^\top : M \in \mathbb{R}^{n\times n}_+,\ M\mathbb{1}_n = M^\top \mathbb{1}_n = \mathbb{1}_n\right\}, \tag{11}$$

where the second equality follows by the Birkhoff-von Neumann theorem. The sequence $\{\mathrm{Perm}_n\}$, viewed as subsets of the consistent sequence in Example 1.5 with $\mathsf{G}_n = \mathsf{S}_n$, is neither intersection- nor projection-compatible. Furthermore, their description (11) is not free because the map $M \mapsto M[1,2,\dots,n]^\top$ is not G_n-equivariant. The smaller descriptions of these permutahedra in [67, 68] are also not free because they are not equivariant. However, there is a sequence of permutahedra arising naturally from a limiting perspective that is both intersection- and projection-compatible and whose description certifies this compatibility.

Fix $q, m \in \mathbb{N}$ and a vector $\lambda \in \mathbb{R}^q$ with distinct entries, and define $\widetilde{\lambda} = (\lambda_1,\dots,\lambda_1,\dots,\lambda_q,\dots,\lambda_q) \in \mathbb{R}^m$ in which λ_i appears m_i times (so $\sum_i m_i = m$). Let $\mathsf{G}_n = \mathsf{S}_{m2^n}$ embedded in G_{n+1} by sending a $m2^n \times m2^n$ matrix $g \in \mathsf{G}_n$ to $g \otimes I_2$. Let $\mathscr{V} = \{\mathbb{R}^{m2^n}\}_n$ with embeddings $x \mapsto x \otimes \mathbb{1}_2$, the normalized inner product $\langle x, y\rangle = (m2^n)^{-1} x^\top y$, and the standard action of G_n. We consider convex hulls of all the vectors in $\mathscr{V}$ containing λ_i in a fraction m_i/m of its entries, which are given using the Birkhoff-von Neumann theorem by

$$\begin{aligned}\mathrm{Perm}(\lambda)_n &= \mathrm{conv}\{g \cdot (\widetilde{\lambda} \otimes \mathbb{1}_{m2^n})\}_{g \in \mathsf{S}_{m2^n}}\\ &= \left\{M\lambda : M \in \mathbb{R}^{m2^n \times q}_+,\ M\mathbb{1}_q = \mathbb{1}_{m2^n},\ M^\top \mathbb{1}_{m2^n} = 2^n[m_1,\dots,m_q]^\top\right\}.\end{aligned} \tag{12}$$

This is an intersection- and projection-compatible sequence of subsets of $\mathscr{V}$. Moreover, the description (12) is free and certifies intersection and projection compatibility. Indeed, let $\mathscr{W} = \mathscr{V}^{\oplus q} = \{\mathbb{R}^{m2^n \times q}\}_n$ and $\mathscr{U} = \mathscr{W} \oplus \mathscr{V}^{\oplus 2} \oplus \mathbb{R}^q$ containing cones $\{\mathbb{R}^{m2^n \times q}_+ \oplus 0 \oplus 0\}$. Then (12) is of the form (ConicSeq) with

$$A_n x = (0, -x, 0, 0),\quad B_n M = (M,\ M\lambda,\ M\mathbb{1}_q,\ (m2^n)^{-1} M^\top \mathbb{1}_{m2^n}),$$

and $u_n = (0, 0, -\mathbb{1}_{m2^n}, -\left[\frac{m_1}{m},\dots,\frac{m_q}{m}\right]^\top)$. Since $\{A_n\}, \{A_n^*\}, \{B_n\}, \{B_n^*\}$ are morphisms and $u_n = u_{n+1}$ under our embedding, Proposition 3.2(a) applies (note that the inner product here is nonstandard, so $A_n^* \neq A_n^\top$ and similarly for B_n).

The insight behind this construction comes from the limiting perspective of Section 3.3. Note that $\mathbb{V}_\infty$ can be viewed as a space of piecewise-constant functions on $[0,1)$. Indeed, define intervals $I_i^{(n)} =$

$[(i-1)/m2^n, i/m2^n)$ and associate to each $v \in \mathbb{V}_n$ the piecewise-constant function $f_v(x) = \sum_{i=1}^{m2^n} v_i \mathbb{1}_{I_i^{(n)}}(x)$ where $\mathbb{1}_S(x) = 1$ if $x \in S$ and 0 otherwise. Note that $v \in \mathbb{V}_n$ and $v \otimes \mathbb{1}_2 \in \mathbb{V}_{n+1}$ define the same function in this way, and that $\langle v, w\rangle = \langle f_v, f_w\rangle_{L^2([0,1))}$. Also, we have for $f \in \mathbb{V}_\infty$

$$(\mathcal{P}_n f)(x) = m2^n \int_{I_i^{(n)}} f \quad \text{if } x \in I_i^{(n)}, \tag{13}$$

hence Jensen's inequality implies $\|\mathcal{P}_n f\|_{L^p} \leq \|f\|_{L^p}$ for all $p \in [1,\infty]$. Thus, we can identify $\mathbb{V}_\infty$ with the space of such step functions, which is contained in $L^p([0,1))$ for all $p \in [1,\infty]$, and we can consider the completion of $\mathbb{V}_\infty$ with respect to any of the L^p norms. If $p < \infty$ we get $\overline{\mathbb{V}_\infty} = L^p([0,1))$ while if $p = \infty$ then $\overline{\mathbb{V}_\infty}$ is the space of functions continuous on $[0,1) \setminus \{m/2^n : m,n \in \mathbb{N}\}$ and having left and right limits everywhere [69, §7.6]. Then $\overline{\mathrm{Perm}(\lambda)_\infty}$ is the closed convex hull of functions in $\overline{\mathbb{V}_\infty}$ that take values λ_i on a subset of $[0,1]$ of measure m_i/m. Furthermore, Theorem 3.6 applies to the description in (12) and yields the following dense subset of $\overline{\mathrm{Perm}(\lambda)_\infty}$.

Proposition 4.1. *Endowing $\mathbb{V}_\infty$ with the $L^p([0,1))$ norm as above for any $p \in [1,\infty]$, we obtain*

$$\begin{aligned}\overline{\mathrm{Perm}(\lambda)_\infty} &= \overline{\mathrm{conv}}\left\{f \in \overline{\mathbb{V}_\infty} : f([0,1)) = \{\lambda_1,\ldots,\lambda_q\},\ |f^{-1}(\lambda_i)| = \frac{m_i}{m}\right\} \\ &= \overline{\left\{\sum_{i=1}^q \lambda_i f_i : f_i \in \overline{\mathbb{V}_\infty},\ f_i \geq 0 \ a.e.\ , \sum_{i=1}^q f_i = 1,\ \int_{[0,1)} f_i = \frac{m_i}{m}\right\}}.\end{aligned} \tag{14}$$

Proof. For the first equality, the inclusion $\subseteq$ is clear. For the reverse inclusion, if $f \in \overline{\mathbb{V}_\infty}$ takes values λ_i on sets $\Omega_i = f^{-1}(\lambda_i)$ of measure m_i/m partitioning $[0,1)$, we can write $f = \sum_{i=1}^q \lambda_i \mathbb{1}_{\Omega_i}$. Then $\mathcal{P}_n f = \sum_{i=1}^q \lambda_i(\mathcal{P}_n \mathbb{1}_{\Omega_i})$, and under the above identification of $\mathbb{R}^{m2^n}$ with piecewise-constant functions, we can view $\mathcal{P}_n \mathbb{1}_{\Omega_i} \in \mathbb{R}_+^{m2^n}$ as nonnegative vectors. The matrix $M = [\mathcal{P}_n\mathbb{1}_{\Omega_1},\ldots,\mathcal{P}_n\mathbb{1}_{\Omega_q}] \in \mathbb{R}_+^{m2^n\times q}$ then satisfies the conditions in (12), hence $\mathcal{P}_n f \in \mathrm{Perm}(\lambda)_n$ and $f = \lim_n \mathcal{P}_n f \in \overline{\mathrm{Perm}(\lambda)_\infty}$ by Lemma 2.22, giving the reverse inclusion.

The second equality follows from Theorem 3.6. Indeed, endow $\mathbb{W}_\infty = \mathbb{V}_\infty^q$ with the norm $\|[f_1,\ldots,f_q]\| = \max_i \|f_i\|_p$ and $\mathbb{U}_\infty$ with the norm $([f_1,\ldots,f_q], g_1, g_2, \mu) = \max\{\|f_i\|_p, \|g_j\|_p, \|\mu\|_\infty\}$. Then $\overline{\mathbb{W}_\infty} = \overline{\mathbb{V}_\infty}^q$ and $\overline{\mathbb{U}_\infty} = \overline{\mathbb{W}_\infty} \oplus \overline{\mathbb{V}_\infty}^2 \oplus \mathbb{R}^q$, and we have $\|A_n\|_{\mathrm{op}} = 1$, $\|B_n\|_{\mathrm{op}} \leq \max\{\sum_i |\lambda_i|, q\}$, and $u_n = u_{n+1}$ for all n. ☐

Schur-Horn orbitopes: We consider the matrix analogs of the above permutahedra, which are convex hulls of all matrices with a given spectrum. Let $\mathsf{G}_n = \mathsf{O}_{m2^n}$ embed in G_{n+1} by sending a $m2^n \times m2^n$ matrix g to $g \otimes I_2$, let $\mathbb{V}_n = \mathbb{S}^{m2^n}$ embed in $\mathbb{V}_{n+1}$ by $X \mapsto X \otimes I_2$ with normalized inner product $\langle X, Y\rangle = (m2^n)^{-1}\mathrm{Tr}(XY)$, and finally let G_n act by conjugation on $\mathbb{V}_n$. Consider the sequence of Schur-Horn orbitopes [70, Eq. (19)]

$$\begin{aligned}\mathrm{SH}(\lambda)_n &= \mathrm{conv}\{g\cdot \mathrm{diag}(\lambda \otimes \mathbb{1}_{2^n})\}_{g \in \mathsf{O}_{m2^n}} \\ &= \left\{\sum_{i=1}^q \lambda_i Y_i : Y_1,\ldots,Y_q \in \mathbb{V}_n \text{ s.t. } \sum_{i=1}^q Y_i = I,\ Y_i \succeq 0,\ \mathrm{Tr}(Y_i) = m_i 2^n \text{ for } i = 1,\ldots,q\right\},\end{aligned} \tag{15}$$

which is the matrix analog of (12). This is again a free description certifying both intersection- and projection-compatibility. Indeed, let $\mathscr{W} = \mathscr{V}^{\oplus q}$ and $\mathscr{U} = \mathscr{W} \oplus \mathscr{V}^{\oplus 2} \oplus \mathbb{R}^q$ containing the cones $\{\mathcal{K}_n = (\mathbb{S}_+^{m2^n})^{\oplus q} \oplus 0 \oplus 0\}$. Then (15) is of the form (ConicSeq) with

$$A_n X = (0, -X, 0, 0), \quad B_n[Y_i]_{i=1}^q = \left([Y_i]_{i=1}^q,\ \sum\nolimits_i \lambda_i Y_i,\ \sum\nolimits_i Y_i,\ (m2^n)^{-1}[\mathrm{Tr}(Y_1),\ldots,\mathrm{Tr}(Y_q)]\right),$$

and $u_n = (0, 0, -I_{m2^n}, -[\frac{m_1}{m},\ldots,\frac{m_q}{m}]^\top)$. Here $\{A_n\},\{A_n^*\},\{B_n\},\{B_n^*\}$ are all morphisms and $u_n = u_{n+1}$ under the above embedding, hence Proposition 3.2(a) applies.

Again, we can use Theorem 3.6 to describe an infinite-dimensional limit $\overline{\mathrm{SH}(\lambda)_\infty}$ of these Schur-Horn orbitopes. We would like to interpret the elements of $\mathrm{SH}(\lambda)_\infty$ as operators with spectral measure $\sum_{i=1}^q \frac{m_i}{m}\delta_{\lambda_i}$, then describe the convex hull of all such operators in $\overline{\mathbb{V}_\infty}$. To that end, we complete $\mathbb{V}_\infty$ with respect to the operator norm and consider spectral measures with respect to the normalized trace τ on $\mathbb{V}_\infty$, which also extends to the limit. Such spectral measures and associated orbitopes exist in more general algebras. In fact, our framework yields descriptions for Schur-Horn orbitopes, as well as a generalization of the Schur-Horn theorem, to self-adjoint elements in so-called approximately finite-dimensional (AF) algebras. These orbitopes generalize both the permutahedra and Schur-Horn orbitopes in this section. We construct these algebras and apply our framework to the resulting orbitopes in Appendix C. In particular, we obtain the following description for the limit in our present setting, proved in Proposition C.2 of that appendix.

Proposition 4.2. *Endow $\mathbb{V}_\infty$ with the operator norm and let $\tau\colon \mathbb{V}_\infty \to \mathbb{R}$ be the normalized trace given by $\tau(X) = (m2^n)^{-1}\mathrm{Tr}(X)$ for $X \in \mathbb{V}_n$. Then τ extends to $\overline{\mathbb{V}_\infty}$, and each $X \in \overline{\mathbb{V}_\infty}$ has a spectral measure μ_X^τ with respect to τ. Furthermore,*

$$\overline{\mathrm{SH}(\lambda)_\infty} = \overline{\mathrm{conv}}\left\{X \in \overline{\mathbb{V}_\infty} : \mu_X^\tau = \sum_{i=1}^q \frac{m_i}{m}\delta_{\lambda_i}\right\} = \overline{\left\{\sum_{i=1}^q \lambda Y_i : Y_i \succeq 0,\ \sum_i Y_i = I,\ \tau(Y_i) = \frac{m_i}{m}\right\}}.$$

Furthermore, the diagonal map $\mathrm{diag}_n\colon \mathbb{S}^{m2^n} \to \mathbb{R}^{m2^n}$ satisfies $\mathrm{diag}_n(\mathrm{SH}(\lambda)_n) = \mathrm{Perm}(\lambda)_n$ by the Schur-Horn theorem. Our limiting descriptions yield the following limiting version of this theorem, proved in greater generality in Proposition C.3 of the above appendix.

Proposition 4.3. *In the setting of Proposition 4.2, we have $\mathrm{diag}(\overline{\mathrm{SH}(\lambda)_\infty}) = \overline{\mathrm{Perm}(\lambda)_\infty}$ where we complete the permutahedra as in Proposition 4.1 with respect to the L^∞ norm.*

Schur-Horn orbitopes are special cases of so-called *spectral polyhedra* studied in [71]. It would be interesting to identify further examples of freely-described sequences of spectral polyhedra arising in applications and to consider their limits.

Remark 4.4. *We remark that the above constructions can be extended to handle vectors and matrices of all sizes, not just of sizes $(m2^n)_{n\in\mathbb{N}}$. This can be done by indexing our consistent sequences by posets. In particular, our results generalize to the following, more complicated, sequences of group representations. If $\mathcal{N}$ is a strict poset, an $\mathcal{N}$-indexed consistent sequence of $\{\mathsf{G}_n\}_{n\in\mathcal{N}}$-representations is a sequence $\{(V_n, \varphi_{N,n})\}_{n<N\in\mathcal{N}}$ of G_n-representations and embeddings $\varphi_{N,n}\colon V_n \hookrightarrow V_N$ for each $n < N$ such that $\varphi_{N,n}$ is G_n-equivariant, and $\varphi_{M,N} \circ \varphi_{N,n} = \varphi_{M,n}$ whenever $n < N < M$. All our results apply in this setting after replacing all occurrences of $n + 1$ by $N > n$. To handle permutahedra and Schur-Horn orbitopes of any sizes we let $\mathcal{N} = \mathbb{N}$ with the divisibility partial order, whereby $n \le m$ iff $m = nk$ for some $k \in \mathbb{N}$, with embeddings $\varphi_{kn,n}\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{nk}$ sending $x \mapsto x \otimes \mathbb{1}_k$ or $\varphi_{kn,n}\colon \mathbb{S}^n \hookrightarrow \mathbb{S}^{nk}$ sending $X \mapsto X \otimes I_k$.*

4.4 Free Spectrahedra

The family of free spectrahedra in Example 1.2(b) may be obtained as a special case of our framework with appropriate selection of description spaces and by imposing compatibility.

Let $\mathscr{V}_0 = \{\mathbb{S}^n\}$ with embeddings by zero-padding and the action of $\mathsf{G}_n = \mathsf{O}_n$ by conjugation. Fix $d, k \in \mathbb{N}$, and let $\mathscr{V} = \mathscr{V}_0^{\oplus d}$, $\mathscr{U} = \mathbb{S}^k \otimes \mathscr{V}_0$, and $\mathscr{W} = \{\mathbb{W}_n = 0\}$.

As the only morphisms $\mathscr{V}_0 \to \mathscr{V}_0$ are multiples of the identity, and elements of $\mathbb{S}^k \otimes \mathbb{S}^n$ can be viewed as $k \times k$ symmetric block matrices with symmetric $n \times n$ blocks of $n \times n$ symmetric matrices, we conclude that the morphisms $\mathscr{V} \to \mathscr{U}$ are precisely maps of the form $(X_1, \ldots, X_d) \mapsto \sum_i L_i \otimes X_i$ for some $L_1, \ldots, L_d \in \mathbb{S}^k$. Note that sequences of adjoints of such maps are also morphisms in this case. As the only G_n-invariants in $\mathbb{S}^n$ are multiples of I_n, the space of freely-described elements in $\mathscr{U}$ is $\{\{L_0 \otimes I_n\}_n : L_0 \in \mathbb{S}^k\}$, which satisfy the condition $L_0 \otimes (I_{n+1} - I_n) \succeq 0$ from Proposition 3.2(a) if and only if $L_0 \succeq 0$. Thus, the parametric family of free descriptions certifying compatibility as in Proposition 3.2(a) is

$$(\mathcal{D}_{\mathcal{L}})_n = \left\{(X_1, \ldots, X_d) \in (\mathbb{S}^n)^d : L_0 \otimes I_n + \sum_{i=1}^d L_i \otimes X_i \succeq 0\right\}, \quad L_0 \succeq 0.$$

which are free spectrahedra parametrized by $\mathcal{L} = (L_0, \ldots, L_d)$. It is common to take either $L_0 = I_k$ (the *monic* case) or $L_0 = 0$ (the *homogeneous* case) [14]. As discussed in Section 1.2, free spectrahedra are fundamental objects in noncommutative free convex and algebraic geometry, see [13, 14] for an introduction.

In particular, they satisfy both intersection and projection compatibility, and more generally closure under so-called *matrix-convex combinations.*

We remark that orthogonal invariance and compatibility alone do not yield closure under matrix-convex combinations. For example, the nuclear norm balls $\{X \in \mathbb{S}^n : \|X\|_* \leq 1\}$ are O_n-invariant and form a compatible sequence which is not matrix-convex. However, a compatible sequence of orthogonally-invariant convex *cones* is indeed matrix convex, as the following result shows.

Proposition 4.5. *Let $\{\mathbb{V}_n = (\mathbb{S}^n)^d\}$ be the consistent sequence of $\{\mathsf{O}_n\}$-representations above, and suppose $\{\mathcal{C}_n \subseteq \mathbb{V}_n\}$ is an intersection- and projection-compatible sequence of convex cones such that $\mathcal{C}_n$ is O_n-invariant for all n. Then $\{\mathcal{C}_n\}$ is matrix-convex.*

Proof. By [72, §2.3], it suffices to show that if $(X_1,\ldots,X_d) \in \mathcal{C}_n$ and $(Y_1,\ldots,Y_d) \in \mathcal{C}_m$ then $(X_1 \oplus Y_1,\ldots,X_d \oplus Y_d) \in \mathcal{C}_{n+m}$, and that $(V^\top X_1 V,\ldots,V^\top X_d V) \in \mathcal{C}_k$ for any isometry $V \in \mathbb{R}^{n\times k}$. For the first, intersection compatibility shows that $(X_1\oplus 0_m,\ldots,X_d\oplus 0_m) \in \mathcal{C}_{n+m}$ and $(Y_1\oplus 0_n,\ldots,Y_d\oplus 0_n) \in \mathcal{C}_{n+m}$. Conjugating the latter tuple by appropriate permutation matrices, we conclude that $(0_n\oplus Y_1,\ldots,0_n\oplus Y_d) \in \mathcal{C}_{n+m}$. Finally, since $\mathcal{C}_{n+m}$ is a convex cone we get $(X_1 \oplus Y_1,\ldots,X_d \oplus Y_d) \in \mathcal{C}_{n+m}$, so $\{\mathcal{C}_n\}$ is closed under direct sums. For the second, any isometry $V \in \mathbb{R}^{n\times k}$ can be written as $V = U\iota_{n,k}$ where $U \in \mathsf{O}_n$ and $\iota_{n,k}\colon \mathbb{R}^k \to \mathbb{R}^n$ is the zero-padding embedding. Observe that $(\iota_{n,k}^\top X_1 \iota_{n,k},\ldots,\iota_{n,k}^\top X_d \iota_{n,k}) = \mathcal{P}_k(X_1,\ldots,X_d)$ is the orthogonal projection of $(X_1,\ldots,X_d)$ onto $\mathbb{V}_k$ when embedded in $\mathbb{V}_n$. Thus, $(V^\top X_1 V,\ldots,V^\top X_d V) = \mathcal{P}_k(U^\top X_1 U,\ldots,U^\top X_d U) \in \mathcal{C}_k$ since $\mathcal{C}_n$ is O_n-invariant and since $\{\mathcal{C}_n\}$ is projection-compatible. ☐

To recap, orthogonal invariance and compatibility of a sequence of convex sets do not yield matrix convexity in general, but they do so for convex cones. With additional choices of description spaces, our framework yields a particular parametric family of matrix-convex sets, namely free spectrahedra.

4.5 Spectral Functions, (Quantum) Entropy, and Variants

Let $\mathscr{V} = \{\mathbb{S}^n\}$ with embeddings by zero-padding and with the action of O_n by conjugation, and let $\mathscr{V}' = \{\mathbb{R}^n\}$ with embeddings by zero-padding and the standard action of S_n. Recall (e.g., [73]) that a convex function $F_n\colon \mathbb{S}^n \to \mathbb{R}$ is O_n-invariant if and only if there exists an S_n-invariant convex function $f_n\colon \mathbb{R}^n \to \mathbb{R}$ satisfying $F_n(X) = f_n(\lambda(X))$ where $\lambda(X) \in \mathbb{R}^n$ is the vector of eigenvalues of $X \in \mathbb{S}^n$. Also, the sequence $\{F_n\colon \mathbb{S}^n \to \mathbb{R}\}$ is intersection-compatible if and only if the sequence $\{f_n\colon \mathbb{R}^n \to \mathbb{R}\}$ is.

Examples of such sequences of functions $\mathfrak{F} = \{F_n\}$ and $\mathfrak{f} = \{f_n\}$ arise in (quantum) information theory, where $\mathfrak{F}$ is the quantum analog of classical information-theoretic parameters $\mathfrak{f}$. These are often intersection-compatible as distributions on n states can be viewed as distributions on $n+1$ states with zero probability on the last state. For example, the negative entropy and relative entropy and their quantum variants are given by

$$\begin{aligned} h_n(x) &= \sum_i x_i \log x_i, & \mathsf{H}_n(X) &= h_n(\lambda(X)) = \mathrm{Tr}(X\log X),\\ D_n(x,y) &= \sum_i x_i \log\frac{x_i}{y_i}, & S_n(X,Y) &= D_n(\lambda(X),\lambda(Y)) = \mathrm{Tr}(X(\log X - \log Y)). \end{aligned} \tag{16}$$

Here $\mathrm{dom}(h_n) = \Delta^{n-1}$ and $\mathrm{dom}(D_n) = (\mathbb{R}^n_+)^2$, while $\mathrm{dom}(\mathsf{H}_n) = \mathcal{D}^{n-1}$ is the spectraplex from Example 1.1(b) and $\mathrm{dom}(S_n) = (\mathbb{S}^n_+)^2$. We use the standard convention that $0\log\frac{0}{y} = 0$ even if $y = 0$, and $x\log\frac{x}{0} = \infty$ when $x \neq 0$ [74, §2.3]. These sequences of functions are intersection-compatible but not projection-compatible (e.g., their domains are not projection-compatible).

Semidefinite approximations: The functions (16) are not semidefinite-representable (i.e., cannot be evaluated using semidefinite programming), though semidefinite approximations of them have been proposed in the literature [36]. We show that these approximations are freely-described, but that these descriptions do not certify intersection compatibility. The family of approximations of [36] to the negative quantum entropy is parametrized by $m, k \in \mathbb{N}$ via their epigraphs

$$E_n^{(m,k)} = \Bigg\{ (X,t) \in \mathbb{S}^n \oplus \mathbb{R} \,\Bigg|\, \exists T_0, \dots, T_m, Z_0, \dots, Z_k \in \mathbb{S}^n \text{ s.t. } Z_0 = I_n,\ \sum_{j=1}^m w_j T_j = -2^{-k} T_0,$$
$$\begin{bmatrix} Z_i & Z_{i+1} \\ Z_{i+1} & X \end{bmatrix} \succeq 0, \text{ for } i = 0, \dots, k-1,\ \begin{bmatrix} Z_k - X - T_j & -\sqrt{s_j} T_j \\ -\sqrt{s_j} T_j & X - s_j T_j \end{bmatrix} \succeq 0,$$
$$\text{for } j = 1, \dots, m,\ \mathrm{Tr}(T_0) \le t \Bigg\},$$

where $s, w \in \mathbb{R}^m$ are the nodes and weights for Gauss-Legendre quadrature.

This is a free description of the form (ConicSeq) which almost, but not quite, satisfies the conditions of Proposition 3.2. Indeed, let $\mathscr{W} = \mathscr{V}^{\oplus(m+k+1)}$ and $\mathscr{U} = \mathscr{V} \oplus (\mathbb{S}^2 \otimes \mathscr{V})^{\oplus(m+k)} \oplus \mathbb{R}$ containing the cones $\{\mathcal{K}_n = \{0\} \oplus (\mathbb{S}^2 \otimes \mathbb{S}^n)_+^{\oplus(m+k)} \oplus \mathbb{R}_+\}$. Define

$$A_n(X,t) = \left(0, \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes X\right)^{\oplus k}, \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes X\right)^{\oplus m}, t\right),$$
$$B_n(T_0, \dots, T_m, Z_1, \dots, Z_k) = \left(2^{-k} T_0 + \sum_{j=1}^m w_j T_j, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes Z_1, \bigoplus_{i=1}^{k-1} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes Z_i + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes Z_{i+1}\right),\right.$$
$$\left.\bigoplus_{j=1}^m \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes Z_k - \begin{bmatrix} 1 & \sqrt{s_j} \\ \sqrt{s_j} & s_j \end{bmatrix} \otimes T_j\right), -\mathrm{Tr}(T_0)\right),$$
$$u_n = \left(0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_n, 0, \dots, 0\right).$$

Note that $\{A_n\}$ and $\{B_n\}$ are morphisms, that $\{u_n\}$ is a freely-described element of $\mathscr{U}$ satisfying $u_{n+1} - u_n \in \mathcal{K}_{n+1}$, but that $\{B_n^*\}$ is *not* a morphism because of the $\mathrm{Tr}(T_0)$ term in B_n. By the proof of Proposition 3.2, this description certifies that $E_n^{(m,k)} \subseteq E_{n+1}^{(m,k)}$ but not $E_{n+1}^{(m,k)} \cap (\mathbb{S}^n \oplus \mathbb{R}) \subseteq E_n^{(m,k)}$. Analogously to the naïve description of the solid simplex in Section 4.1, this free description does not make the intersection compatibility of $\{E_n^{(m,k)}\}$ obvious.

Parametric families: We can use the description spaces of [36] above to derive parametric families of freely-described sets. When compatibility is not required, the resulting family includes the approximation $\{E_n^{(m,k)}\}_n$ of [36]; when compatibility is imposed, we obtain a smaller family excluding $\{E_n^{(m,k)}\}_n$.

Note that $\mathbb{S}^2 \otimes \mathbb{S}^n \cong (\mathbb{S}^n)^3$ as O_n-representations, and these isomorphisms commute with zero-padding, so that $\mathscr{U} \cong \mathscr{V}^{1+3(m+k)} \oplus \mathbb{R}$ as consistent sequences. As $\dim(\mathbb{S}^n)^{\mathsf{O}_n} = 1$ and $\dim \mathcal{L}(\mathbb{S}^n)^{\mathsf{O}_n} = 2$, the dimension of invariants parametrizing free descriptions are

$$\dim \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{O}_n} = 3[2(m+k)+1], \quad \dim \mathcal{L}(\mathbb{W}_n, \mathbb{U}_n)^{\mathsf{O}_n} = 3(m+k+1)[2(m+k)+1],$$
$$\dim \mathbb{U}_n^{\mathsf{O}_n} = 2 + 3(m+k).$$

When $m = k = 3$ (the default values in the implementation of [36]), we get

$$\dim \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{O}_n} = 39, \quad \dim \mathcal{L}(\mathbb{W}_n, \mathbb{U}_n)^{\mathsf{O}_n} = 273, \quad \dim \mathbb{U}_n^{\mathsf{O}_n} = 20.$$

As the only morphisms of sequences $\mathscr{V} \to \mathscr{V}$ are multiples of the identity, and the only morphisms $\mathscr{V} \to \mathbb{R}$ are multiples of the trace, the dimensions of $\{A_n\}, \{B_n\}$ satisfying the conditions of Proposition 3.2(a) are

$$\dim \Big\{ \{A_n \colon \mathbb{V}_n \to \mathbb{U}_n\} \text{ morphism} \Big\} = 3(m+k) + 2 = 20,$$
$$\dim \Big\{ \{B_n \colon \mathbb{W}_n \to \mathbb{U}_n\} : \text{both } \{B_n\} \text{ and } \{B_n^*\} \text{ morphisms} \Big\} = (m+k+1)[3(m+k)+1] = 133.$$

4.6 Graph Parameters

Let $\mathscr{V} = \{\mathbb{S}^n\}$ with embeddings by zero-padding and the action of $\mathsf{G}_n = \mathsf{S}_n$ by conjugation. As discussed in Section 1.1.2 and Example 1.7, graph parameters are sequences $\{f_n \colon \mathbb{S}^n \to \mathbb{R}\}$ of G_n-invariant functions, and many standard graph parameters are convex (or concave) and satisfy either intersection or projection compatibility. Furthermore, it is desirable to find relaxations for standard graph parameters satisfying the same compatibility conditions. We consider here the examples of max-cut and inverse stability number, and we show that their usual relaxations are freely-described with descriptions certifying their compatibility. We then derive parametric families of free descriptions from our framework that may be used to fit graph parameters to data.

Max-cut: Computing the max-cut value of a weighted undirected graph amounts to evaluating the support function of the *cut polytope* $\mathrm{CUT}_n = \mathrm{conv}\{xx^\top : x \in \{\pm 1\}^n\}$. The sequence of cut polytopes $\{\mathrm{CUT}_n\}$ viewed as subsets of $\mathscr{V}$ is projection-compatible and compact, hence the sequence of their support functions and the max-cut value itself are intersection-compatible (see Section 1.3). Approximation of the max-cut value reduces to approximation of the cut polytopes. A standard outer approximation of the sequence of cut polytopes is the sequence of elliptopes

$$\mathcal{E}_n = \{X \in \mathbb{S}^n : X \succeq 0, \mathrm{diag}(X) = \mathbb{1}_n\}. \tag{17}$$

The sequence $\{\mathcal{E}_n\}$ also satisfies projection compatibility, and the above is a free description certifying this compatibility. Indeed, let $\mathscr{W} = \{0\}$ and $\mathscr{U} = \mathscr{V} \oplus \{\mathbb{R}^n\}$ where the latter sequence is the usual one from Example 1.5, with cones $\mathscr{K} = \{\mathbb{S}^n_+ \oplus \{0\}\}$. Then (17) is of the form (ConicSeq) with $A_nX = (X, \mathrm{diag}(X))$, $B_n = 0$, and $u_n = (0, -\mathbb{1}_n)$. Note that $\{A_n\}, \{A_n^*\}$ are morphisms and $u_{n+1} - u_n = (0, -e_{n+1}) = (e_{n+1}e_{n+1}^\top, 0) - A_{n+1}e_{n+1}e_{n+1}^\top$, hence Proposition 3.2(b) applies. Neither the cut polytopes nor the elliptopes is intersection-compatible, as zero-padding a matrix with all-1's diagonal does not yield such a matrix. The sequences $\{\mathrm{CUT}_n - I_n\}$ and $\{\mathcal{E}_n - I_n\}$ are, however, both intersection- and projection-compatible, and the shifted elliptopes admit free descriptions certifying their compatibility.

Inverse stability number: Computing the inverse stability number reduces to evaluating the support functions of $\mathcal{D}_n = \mathrm{conv}\{xx^\top : x \in \Delta^{n-1}\}$, see [75]. A natural SDP relaxation for this problem is evaluating the support function of

$$\widetilde{\mathcal{D}}_n = \{X \in \mathbb{S}^n : X \succeq 0, X \geq 0, \mathbb{1}_n^\top X \mathbb{1}_n = 1\},$$

where $X \geq 0$ denotes an entrywise nonnegative matrix. Both $\{\mathcal{D}_n\}$ and $\{\widetilde{\mathcal{D}}_n\}$ are intersection-compatible. Moreover, the above description of $\{\widetilde{\mathcal{D}}_n\}$ is free and certifies this compatibility as in Proposition 3.2(b). Indeed, let $\mathscr{W} = \{0\}$ and $\mathscr{U} = \mathscr{V} \oplus \mathbb{R}$ with cones $\mathcal{K}_n = (\mathbb{S}^n_+ \cap \mathbb{R}^{n\times n}_+) \oplus 0$. Then the above description of $\widetilde{\mathcal{D}}_n$ is of the form (ConicSeq) with $A_nX = (X, \mathbb{1}_n^\top X \mathbb{1}_n)$, $B_n = 0$, and $u_n = (0, -1)$. Note that $\{A_n\}$ is a morphism and $u_n = u_{n+1}$, hence Proposition 3.2(a) applies. Neither $\mathcal{D}_n$ nor $\widetilde{\mathcal{D}}_n$ is projection-compatible since their support functions, which are the inverse stability number and its semidefinite approximation above, are not intersection compatible (see Section 1.3). Indeed, appending isolated vertices to a graph increases its stability number.

Parametric families: Beyond the two particular preceding examples, we derive here an expressive parametric family of convex graph parameters; in addition to above examples, our family also includes several other examples from [76]. To that end, let $\mathscr{W} = \mathscr{U} = \mathrm{Sym}^2(\mathrm{Sym}^{\leq 2}\{\mathbb{R}^n\}) = \{\mathbb{S}^{\binom{n+2}{2}}\}$. The dimensions of invariants parametrizing free descriptions in this case is too large for us to write explicit bases for them as functions of n. Instead, we compute these dimensions using the algorithm in Section 6.1 below, see Example 6.2(b):

$$\dim \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n} = 93, \quad \dim \mathcal{L}(\mathbb{W}_n, \mathbb{U}_n)^{\mathsf{G}_n} = 1068, \quad \dim \mathbb{W}_n^{\mathsf{G}_n} = 17, \quad \text{for all } n \geq 8.$$

Using the same algorithm, the dimensions of sequences $\{A_n\}, \{B_n\}$ certifying intersection compatibility as in Proposition 3.2(a) are

$$\begin{aligned} &\dim \Big\{\{A_n \colon \mathbb{V}_n \to \mathbb{U}_n\} \text{ morphism}\Big\} = 19, \\ &\dim \Big\{\{B_n \colon \mathbb{W}_n \to \mathbb{U}_n\} : \text{both } \{B_n\} \text{ and } \{B_n^*\} \text{ morphisms}\Big\} = 104. \end{aligned} \tag{18}$$

$$X = \begin{bmatrix} 0 & 0.75 & 0 & 0 & 0.25 & 0 \\ 0.75 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.25 & 1 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0.75 \\ 0.25 & 0 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.75 & 0.5 & 0 \end{bmatrix}$$

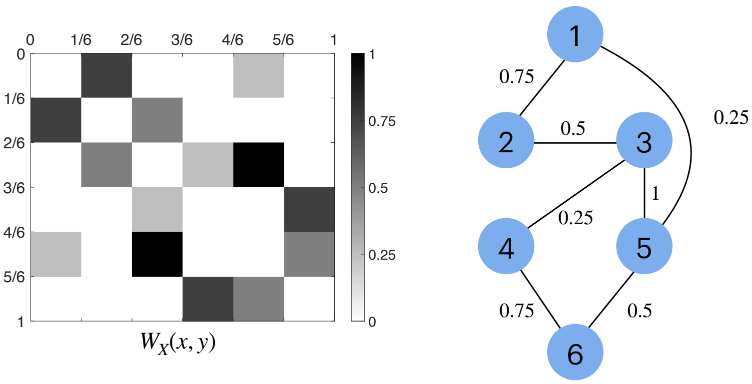


Figure 2: Weighted undirected graph represented as a graph, an adjacency matrix X, and a symmetric function (graphon) W_X on $[0,1]^2$.

The algorithm we present in Section 6 further allows us to compute bases for these invariants in a fixed dimension and extend them to any other; in turn, these are useful for fitting graph parameters defined for graphs of all sizes given data.

4.7 Graphon Parameters

A different embedding between graphs arises in the theory of graphons [15], where a weighted graph $X \in \mathbb{S}^{2^n}$ is viewed as a step function $W_X\colon [0,1]^2 \to \mathbb{R}$ defined by $W_X(x,y) = X_{i,j}$ if $(x,y) \in [(i-1)/2^n, i/2^n) \times [(j-1)/2^n, j/2^n)$; see Figure 2. Note that X and $X \otimes \mathbb{1}_2\mathbb{1}_2^\top \in \mathbb{S}^{2^{n+1}}$ correspond to the same step function, and that the inner product of two such step functions W_X, W_Y in $L^2([0,1]^2)$ equals the normalized Frobenius inner product $\langle X, Y\rangle = 2^{-2n}\mathrm{Tr}(X^\top Y)$. We therefore define the graphon consistent sequence $\mathscr{V} = \{\mathbb{V}_n = \mathbb{S}^{2^n}\}$ with embeddings $\varphi_n(X) = X \otimes \mathbb{1}_{2\times 2}$, the above normalized inner products, and the action of $\mathsf{G}_n = \mathsf{S}_{2^n}$ by conjugation. Here G_n is embedded into G_{n+1} by sending a permutation matrix g to $g \otimes I_2$. We can extend our consistent sequence to include symmetric matrices of any size by having our consistent sequence be indexed by the poset $\mathbb{N}$ with the divisibility partial order, see Remark 4.4. The graphon sequence is finitely-generated, as the following computer-assisted proof shows.

Proposition 4.6. *The graphon sequence $\{\mathbb{V}_n = \mathbb{S}^{2^n}\}$ is generated in degree* 2.

Proof. Define $E_1^{(n)} = e_1^{(2^n)}(e_1^{(2^n)})^\top$ and $E_2^{(n)} = e_1^{(2^n)}(e_2^{(2^n)})^\top + e_2^{(2^n)}(e_1^{(2^n)})^\top$, whose G_n-orbits span $\mathbb{V}_n$. We verify computationally that $\dim \sum_{i=1}^2 \mathbb{R}[\mathsf{S}_{2^3}]\varphi_2(E_i^{(2)}) = \dim \mathbb{V}_3$, see the GitHub repository. Therefore,

$$E_i^{(3)} = \sum\nolimits_{j=1}^{2} r_{i,j}\varphi_2(E_j^{(2)}), \qquad \text{for } i \in [2],\ r_{i,j} \in \mathbb{R}[S_{2^3}]. \tag{19}$$

Let $\psi_n(X) = X \oplus 0$ be an embedding of $\mathbb{V}_n$ into $\mathbb{V}_{n+1}$ by zero-padding, and note that $E_i^{(n+1)} = \psi_n(E_i^{(n)})$ and that ψ_n commutes with a different embedding of S_{2^n} into $\mathsf{S}_{2^{n+1}}$, namely, one that sends $g \mapsto g \oplus I_{2^n}$. Applying ψ_n to (19), we conclude that $E_i^{(n+1)}$ can be written as $\mathbb{R}[\mathsf{S}_{2^{n+1}}]$-linear combinations of $\varphi_n(E_j^{(n)})$ for all $n \geq 2$, hence that $\mathbb{R}[\mathsf{S}_{2^{n+1}}]\varphi_n(\mathbb{V}_n) = \mathbb{V}_{n+1}$ for all $n \geq 2$. □

Graphon parameters: A permutation-invariant and intersection-compatible sequence of functions $\mathfrak{f} = \{f_n\colon \mathbb{V}_n \to \mathbb{R}\}$ is called a *graphon parameter*, since these are precisely the functions of graphs that do not depend on their inputs' vertex labels, and that only depend on their input graphs via their associated graphons. A family of graphon parameters that plays a central role in the theory of graphons and in extremal combinatorics are graph homomorphism densities [15]. Their convexity is related to weakly-norming graphs and Sidorenko's conjecture, a major open problem in extremal combinatorics [77, 78].

We can obtain parametric families of graphon parameters by taking the gauge functions of parametric families of intersection-compatible and freely-described convex sets. For example, let $\mathscr{U} = \mathbb{S}^k \otimes \mathscr{V}$ with cones

$\mathscr{K} = \{\mathcal{K}_n = (\mathbb{S}^k \otimes \mathbb{S}^{2^n})_+\}$ and $\mathscr{W} = \{\mathbb{W}_n = 0\}$. Using Proposition 3.2(a), we get the following parametric family of freely-described and intersection-compatible sets, parametrized by $L_1, \ldots, L_7 \in \mathbb{S}^k$:

$$\mathcal{C}_n = \Bigg\{ X \in \mathbb{S}^{2^n} : \frac{\mathbb{1}^\top X \mathbb{1}}{2^{2n}} L_1 \otimes \mathbb{1}\mathbb{1}^\top + \frac{\mathrm{Tr}(X)}{2^n} L_2 \otimes \mathbb{1}\mathbb{1}^\top + L_3 \otimes \frac{1}{2^n}\left(X\mathbb{1}\mathbb{1}^\top + \mathbb{1}\mathbb{1}^\top X\right) \\ + L_4 \otimes \left(\mathrm{diag}(X)\mathbb{1}^\top + \mathbb{1}\mathrm{diag}(X)^\top\right) + L_5 \otimes X + L_6 \otimes \mathbb{1}\mathbb{1}^\top + L_7 \otimes (2^n I_{2^n}) \succeq 0 \Bigg\}, \quad L_7 \succeq 0. \tag{20}$$

Note that all the functions of X appearing in the above description only depend on the associated step graphon W_X. For example, $\frac{\mathbb{1}^\top X \mathbb{1}}{2^{2n}} = \int_{[0,1]^2} W_X(t,s)\,\mathrm{d}t\,\mathrm{d}s$ and $\frac{\mathrm{Tr}(X)}{2^n} = \int_0^1 W_X(t,t)\,\mathrm{d}t$.

Expressing the above free description in terms of graphons yields a description of a limiting convex set by Theorem 3.6. Its gauge function is, in turn, a convex graphon parameter extending continuously to the corresponding limit. Endow $\mathbb{V}_\infty$ with the $L^\infty([0,1]^2)$-norm, noting that $\|\mathcal{P}_n W\|_\infty \le \|W\|_\infty$ for all $W \in L^\infty([0,1]^2)$ by Jensen's inequality. We view elements $[W_{i,j}(x,y)]_{i,j=1}^k \in \mathbb{S}^k \otimes L^\infty([0,1]^2)$ as symmetric matrices with entries in $L^\infty([0,1]^2)$ (so $W_{j,i} = W_{i,j}$), and denote

$$[W_{i,j}(x,y)]_{i,j=1}^k \succeq 0 \quad \text{if} \quad \int_{[0,1]^2} \sum_{i,j=1}^k W_{i,j}(x,y) f_i(x) f_j(y)\, dx\, dy \ge 0 \quad \text{for all} \quad f_1, \ldots, f_k \in L^2([0,1]). \tag{21}$$

Remarkably, convex graphon parameters that extend continuously to this limit are also projection-compatible by [79, Thm. 3.17].

Proposition 4.7. *If $L_7 = 0$ in (20) then*

$$\overline{C_\infty} = \Bigg\{ W \in \overline{\mathbb{V}_\infty} : \overline{A_\infty}(W) + u_\infty := \Big[(L_1)_{i,j} \int_{[0,1]^2} W(s,t)\, ds\, dt + (L_2)_{i,j} \int_{[0,1]} W(t,t)\, dt \\ + (L_3)_{i,j} \int_{[0,1]} [W(x,t) + W(t,y)]\, dt + (L_4)_{i,j}[W(x,x) + W(y,y)] + (L_5)_{i,j} W(x,y)\Big]_{i,j=1}^k + L_6 \succeq 0 \Bigg\},$$

where $\overline{\mathbb{V}_\infty} \subseteq L^\infty([0,1]^2)$ is a subspace of symmetric bounded measurable functions on $[0,1]^2$.

Proof. Endow $\mathbb{U}_\infty = \mathbb{S}^k \otimes \mathbb{V}_\infty$ with the norm $\|[W_{i,j}]_{i,j=1}^k\| = \max_{i,j} \|W_{i,j}\|_\infty$, so $\overline{\mathbb{U}_\infty} = \mathbb{S}^k \otimes \overline{\mathbb{V}_\infty}$. Then $\overline{\mathcal{K}_\infty}$ is the cone of PSD matrices $[W_{i,j}(x,y)]_{i,j=1}^k \succeq 0$ in $\overline{\mathbb{U}_\infty}$. Furthermore, for any $W \in \mathbb{V}_\infty$ we have

$$\|A_\infty\| \le \max_{i,j} \Big(|(L_1)_{i,j}| + |(L_2)_{i,j}| + 2|(L_3)_{i,j}| + 2|(L_4)_{i,j}| + |(L_5)_{i,j}| \Big) \|W\|_\infty,$$

so A_∞ extends continuously to the limit, and $u_\infty = L_6 \in U_1 \subseteq \overline{\mathbb{U}_\infty}$ satisfies $\mathcal{P}_n u_\infty = L_6$ for all n. Thus, Theorem 3.6 yields the claim. ☐

We require $L_7 = 0$ since there is no $u_\infty \in \overline{\mathbb{V}_\infty}$ satisfying $\mathcal{P}_n u_\infty = 2^n I_{2^n}$ for all n.

4.8 Compatibility in Inverse Problems

Our compatibility conditions naturally arise in the context of inverse problems, where we demonstrate that it is desirable to use regularizers which are both intersection- and projection-compatible.

Consider a consistent sequence $\mathscr{V} = \{\mathbb{V}_n\}$ of $\{\mathsf{G}_n\}$-representations. A popular approach to recover $x \in \mathbb{V}_n$ from $m \in \mathbb{N}$ linear observations takes as input a forward map $A\colon \mathbb{V}_n \to \mathbb{R}^m$ and data $y \in \mathbb{R}^m$ and outputs

$$F_{m,n}(A,y) = \underset{x \in \mathbb{V}_n}{\mathrm{argmin}}\ f_n(x) + \lambda \|Ax - y\|_2^2, \tag{22}$$

where $f_n\colon \mathbb{V}_n \to \mathbb{R}$ is a convex regularizer promoting desired structure in the solution. It is desirable for the maps $F_{m,n}$ defined in (22)—which can be instantiated for any $(A,y) \in \mathcal{L}(\mathbb{V}_n, \mathbb{R}^m) \oplus \mathbb{R}^m$ and for any $n, m \in \mathbb{N}$—to satisfy

$$F_{m,n+1}(A\mathcal{P}_n, y) = F_{m,n}(A,y), \tag{23}$$

whenever the corresponding minimizers are unique. Indeed, condition (23) requires the recovered solution to lie in $\mathbb{V}_n$ if the data only depends on the component of $x \in \mathbb{V}_{n+1}$ in $\mathbb{V}_n$, as this property avoids overfitting. Condition (23) holds when the sequence of regularizers is both intersection and projection-compatible. Indeed, if the sequence of regularizers $\mathfrak{f} = \{f_n\}$ is *projection-compatible*, then

$$\min_{\widetilde{x}\in\mathbb{V}_{n+1}} f_{n+1}(\widetilde{x}) + \lambda\|A\mathcal{P}_n\widetilde{x} - b\|_2^2 = \min_{x\in\mathbb{V}_n} \min_{\substack{\widetilde{x}\in\mathbb{V}_{n+1}\\ \mathcal{P}_n\widetilde{x}=x}} f_{n+1}(\widetilde{x}) + \lambda\|Ax - y\|_2^2 = \min_{x\in\mathbb{V}_n} f_n(x) + \lambda\|Ax - y\|_2^2$$

Moreover, if $x_* = F_{m,n}(A, y)$ minimizes $f_n(x) + \lambda\|Ax - y\|_2^2$ and $\mathfrak{f}$ is *intersection-compatible*, then $f_n(x_*) + \lambda\|Ax_* - y\|_2^2 = f_{n+1}(x_*) + \lambda\|A\mathcal{P}_n x_* - y\|_2^2$ and hence $x_* = F_{m,n+1}(A\mathcal{P}_n, y)$, showing (23).

5 Constant-Sized Invariant Conic Programs

In the previous sections, we studied freely-described sequences of convex sets $\{\mathcal{C}_n\}$ contained in a consistent sequence $\mathscr{V}$. These convex sets are group-invariant whenever the cones $\mathcal{K}_n$ in their descriptions (ConicSeq) are group-invariant, which is the case for all the standard sequences of cones we consider. In this section, we further consider optimizing sequences of invariant linear functionals over such sequences of sets. Each program in the sequence can be simplified by restricting its domain to invariant vectors [24, §3]. As we have seen in Proposition 2.3, when $\mathscr{V}$ is finitely-generated the dimensions of its spaces of invariants stabilize, so the size of the variables in such programs stabilizes as well. However, the size of the constraints may not stabilize, because the invariant sections of the cones $\{\mathcal{K}_n^{\mathsf{G}_n}\}$ may grow in complexity. For example, if $\mathcal{K}_n$ is the cone of n-variate degree k polynomials that are nonnegative over all of $\mathbb{R}^n$ and $\mathsf{G}_n = \mathsf{S}_n$, the best-known description of $\mathcal{K}_n^{\mathsf{G}_n}$ has complexity which is a (nonconstant) polynomial in n [80, 81, 82]. We therefore seek conditions for the existence of *constant-sized* descriptions for $\{\mathcal{K}_n^{\mathsf{G}_n}\}$, and bounds on the value of n after which the size stabilizes in the sense of Definition 1.10. Constant-sized descriptions for symmetric PSD and relative entropy cones have been obtained on a case-by-case basis in the literature [25, 3, 26, 27]. In this section, we explain how these results can be generalized and derived systematically from an interplay between representation stability and the structure of the cones in question.

5.1 The PSD Cone and Variants

We begin by giving constant-sized descriptions for certain sequences of PSD cones. We do so by deriving constant-sized bases for spaces of invariants in terms of which membership in the cones is simply expressed. The following is a precise statement of Theorem 1.11 stated informally in Section 1.

Theorem 5.1. *Let $\mathscr{V}_0 = \{\mathbb{R}^n\}$ with $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, or S_n as in Example 1.5, and let $\mathscr{V} = \{\mathbb{V}_n\}$ be a $\mathscr{V}_0$-module generated in degree d and presented in degree k. Then the cones $\{\mathrm{Sym}^2_+(\mathbb{V}_n)^{\mathsf{G}_n}\}$ admit constant-sized descriptions for $n \geq k + d$.*

Proof. By Theorem 2.16, there exists a finite set Λ satisfying

$$\mathbb{V}_n = \bigoplus_{\lambda\in\Lambda} \underbrace{\mathbb{W}_{\lambda[n]}^{m_\lambda}}_{=:\mathbb{V}_{\lambda[n]}}$$

where $\mathbb{W}_{\lambda[n]}$ is a G_n-irreducible and $\mathbb{V}_{\lambda[n]}$ is the corresponding isotypic component. Invariant elements of $\mathbb{U}_n = \mathrm{Sym}^2(\mathbb{V}_n)$ are equivariant and self-adjoint endomorphisms of $\mathbb{V}_n$. If $X \in \mathbb{U}_n^{\mathsf{G}_n}$ is such an endomorphism and $\lambda \neq \mu \in \Lambda$ index distinct irreducibles, then $\mathcal{P}_{\mathbb{V}_{\mu[n]}} X|_{\mathbb{V}_{\lambda[n]}} = 0$ by Schur's Lemma [62, §1.2]. Because the irreducibles of $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, and S_n are of real type [83] (meaning they remain irreducible when complexified), Schur's Lemma also implies that $\mathcal{P}_{\mathbb{W}_{\lambda[n]}} X|_{\mathbb{W}_{\lambda[n]}}$ is a multiple of the identity for each $\lambda \in \Lambda$, hence $\mathcal{P}_{\mathbb{V}_{\lambda[n]}} X|_{\mathbb{V}_{\lambda[n]}} = X_\lambda \otimes I_{\dim \mathbb{W}_{\lambda[n]}}$ for some $X_\lambda \in \mathbb{S}^{m_\lambda}$. We conclude that there exists an orthogonal matrix Q_n depending on the irreducible decomposition of $\mathbb{V}_n$ satisfying

$$\mathbb{U}_n^{\mathsf{G}_n} = \left\{ Q_n \bigoplus_{\lambda\in\Lambda} (X_\lambda \otimes I_{\dim \mathbb{W}_{\lambda[n]}}) Q_n^* : X_\lambda \in \mathbb{S}^{m_\lambda} \right\}, \tag{24}$$

hence

$$\mathrm{Sym}^2_+(\mathbb{V}_n)^{\mathsf{G}_n} = \{X \in \mathbb{U}_n^{\mathsf{G}_n} : X \succeq 0\} = \left\{ Q_n \bigoplus_{\lambda\in\Lambda_n} (X_\lambda \otimes I_{\dim \mathbb{W}_\lambda}) Q_n^* : X_\lambda \in \mathbb{S}_+^{m_\lambda} \right\}. \tag{25}$$

Thus, we obtain constant-sized descriptions by defining $\mathbb{U} = \bigoplus_{\lambda\in\Lambda} \mathbb{S}^{m_\lambda}$ and $T_n \colon \mathbb{U} \to \mathbb{U}_n^{\mathsf{G}_n}$ sending $(X_\lambda)_{\lambda\in\Lambda}$ to $Q_n \bigoplus_{\lambda\in\Lambda}(X_\lambda \otimes I_{\dim \mathbb{W}_\lambda})Q_n^*$, which maps $\mathcal{K} = \bigoplus_{\lambda\in\Lambda} \mathbb{S}_+^{m_\lambda}$ onto $\mathrm{Sym}^2_+(\mathbb{V}_n)^{\mathsf{G}_n}$. □

We now instantiate $\mathscr{V}$ to obtain more concrete corollaries.

Corollary 5.2. *If $\mathsf{G}_n = \mathsf{S}_n, \mathsf{D}_n$ or B_n acts on $\mathbb{R}^n$ as in Example 1.5, then the cones $\mathrm{Sym}^2_+(\mathrm{Sym}^{\leq d}\mathbb{R}^n)^{\mathsf{G}_n} \cong \left(\mathbb{S}_+^{\binom{n+k}{k}}\right)^{\mathsf{G}_n}$ admit constant-sized descriptions by (25) for $n \geq 2d$ if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ and $n \geq 2d+1$ if $\mathsf{G}_n = \mathsf{D}_n$.*

Proof. The sequence $\mathscr{V} = \mathrm{Sym}^{\leq d}\mathscr{V}_0$ is generated in degree d and presented in degree d if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ or in degree $d+1$ if $\mathsf{G}_n = \mathsf{D}_n$, by Theorem 2.11 and Example 2.8. □

To obtain constant-sized descriptions for cones of invariant sums-of-squares, we consider equivariant images of the PSD cones above.

Proposition 5.3. *Let $\mathscr{U} = \{\mathbb{U}_n\}$ and $\mathscr{W} = \{\mathbb{W}_n\}$ be sequences of $\{\mathsf{G}_n\}$-representations (not necessarily consistent). Let $\{\mathcal{K}_n \subseteq \mathbb{U}_n\}$ be a sequence of convex cones such that $\mathcal{K}_n$ is G_n-invariant for each n. If $\{\mathcal{K}_n^{\mathsf{G}_n}\}$ admits constant-sized descriptions for $n \geq t$, then so does $\{\pi_n(\mathcal{K}_n \cap \mathbb{L}_n)^{\mathsf{G}_n} \subseteq \mathbb{W}_n\}$ for any sequence $\{\pi_n \in \mathcal{L}(\mathbb{U}_n, \mathbb{W}_n)^{\mathsf{G}_n}\}$ and any sequence of G_n-invariant subspaces $\mathbb{L}_n \subseteq \mathbb{U}_n$.*

Proof. Suppose $\mathcal{K}_n^{\mathsf{G}_n} = T_n(\mathcal{K} \cap \mathbb{L}_n')$ for $n \geq t$ where $T_n \colon \mathbb{U} \to \mathbb{U}_n^{\mathsf{G}_n}$ and $\mathbb{L}_n' \subseteq \mathbb{U}$ are subspaces as in Definition 1.10. Because π_n is G_n-equivariant,

$$\pi_n(\mathcal{K}_n \cap \mathbb{L}_n)^{\mathsf{G}_n} = \pi_n(\mathcal{K}_n^{\mathsf{G}_n} \cap \mathbb{L}_n^{\mathsf{G}_n}) = (\pi_n \circ T_n)(\mathcal{K} \cap \mathbb{L}_n' \cap T_n^{-1}(\mathbb{L}_n^{\mathsf{G}_n})).$$

Noting that $\mathbb{L}_n' \cap T_n^{-1}(\mathbb{L}_n^{\mathsf{G}_n})$ is a subspace of $\mathbb{U}$, we get the desired constant-sized descriptions. □

We now prove Theorem 1.12 giving constant-sized descriptions for invariant sums-of-squares.

Proof (Theorem 1.12). If $v(x)$ is the vector whose coordinates are all the monomials in the $\binom{n}{k}$ variables $x_{i_1,\ldots,i_k}$ of degree at most d, then [64, Thm. 3.39] yields

$$\mathrm{SOS}_{\mathbb{U}_n} = \pi_n\left(\mathrm{Sym}^2_+\left(\mathrm{Sym}^{\leq d}\left(\bigwedge^k \mathbb{R}^n\right)\right)\right), \quad \pi_n(M) = v(x)^\top M v(x) + \mathcal{I}_n.$$

The map $\pi_n \colon \mathrm{Sym}^2\left(\mathrm{Sym}^{\leq d}\left(\bigwedge^k \mathbb{R}^n\right)\right) \to \mathbb{U}_n$ is equivariant by definition of the action of G_n and the invariance of $\mathcal{I}_n$. If $\mathscr{V}_0 = \{\mathbb{R}^n\}$ as in Example 1.5, then $\mathscr{V} = \mathrm{Sym}^{\leq d}\left(\bigwedge^k \mathscr{V}_0\right)$ is a $\mathscr{V}_0$-module generated in degree kd, and presented in degree kd if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ and in degree $kd+1$ if $\mathsf{G}_n = \mathsf{D}_n$ by Theorem 2.11. Thus, the result follows from Theorem 5.1 and Proposition 5.3. □

Note that Theorem 1.12 applies to *any* sequence of invariant ideals, not necessarily related to each other across dimensions, so that $\{\mathbb{U}_n\}$ in this result is not necessarily a consistent sequence. Nevertheless, it is a sequence of equivariant images of a consistent sequence, a fact crucial to the proof.

5.2 The Relative Entropy Cone and Variants

For a finite set $\mathcal{A}$, define the associated relative entropy cone

$$\mathrm{RE}_{\mathcal{A}} = \{(\nu, c, t) \in \mathbb{R}^{\mathcal{A}} \oplus \mathbb{R}^{\mathcal{A}} \oplus \mathbb{R} : \nu, c \geq 0,\ D(\nu, c) \leq t\}, \tag{26}$$

where $D(\nu, c) = \sum_{\alpha\in\mathcal{A}} \nu_\alpha \log\left(\frac{\nu_\alpha}{c_\alpha}\right)$ is the relative entropy.

Proposition 5.4. *Let $\mathscr{V}_0$ be a consistent sequence of $\{\mathsf{G}_n\}$ representations and $\mathscr{V} = \{\mathbb{R}^{\mathcal{A}_n}\}$ be a permutation $\mathscr{V}_0$-module for finite $\mathcal{A}_n$ satisfying $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ (Definition 2.14). If $\dim \mathbb{V}_n^{\mathsf{G}_n}$ is constant for all $n \geq d$, then the invariant section $\{\mathrm{RE}_{\mathcal{A}_n}^{\mathsf{G}_n}\}$ of the cones (26) admit constant-sized descriptions for $n \geq d$.*

Proof. Let $k = \dim(\mathbb{R}^{\mathcal{A}_d})^{\mathsf{G}_d}$ and fix $n \geq d$. Let $\{\alpha_j\}_{j\in[k]} \subseteq \mathcal{A}_n$ be a set of G_n-orbit representatives, and let $\mathbb{1}_{j,n} = \sum_{g\in \mathsf{G}_n/\mathsf{Stab}_{\mathsf{G}_n}(\alpha_j)} e_{g\alpha_j}$ for each $j \in [k]$, so that $\{\mathbb{1}_{j,n}\}_{j\in[k]}$ is a basis for $(\mathbb{R}^{\mathcal{A}_n})^{\mathsf{G}_n} \cong \mathbb{R}^k$. Let $\mathscr{U} = \mathscr{V}^{\oplus 2} \oplus \mathbb{R} = \{\mathbb{U}_n\}$, which contains the relevant relative entropy cones $\{\mathrm{RE}_{\mathcal{A}_n}\}$. Then a basis for $\mathbb{U}_n^{\mathsf{G}_n}$ consists of $\{(\mathbb{1}_{j,n}, 0, 0), (0, \mathbb{1}_{j,n}, 0)\}_{j\in[k]} \cup \{(0, 0, 1)\}$.

If $(\nu, c, t) \in \mathbb{U}_n^{\mathsf{G}_n}$ for $n \geq d$ is expanded as $\nu = \sum_{j=1}^k \hat{\nu}_j \mathbb{1}_{j,n}$ and similarly for c, then

$$\mathrm{RE}_{\mathcal{A}_n}^{\mathsf{G}_n} = \left\{ (\nu, c, t) \in \mathbb{U}_n^{\mathsf{G}_n} : \hat{\nu}, \hat{c} \geq 0,\ \sum_{j=1}^k |\mathsf{G}_n/\mathsf{Stab}_{\mathsf{G}_n}(\alpha_j)| \hat{\nu}_j \log\left(\frac{\hat{\nu}_j}{\hat{c}_j}\right) \leq t \right\}.$$

Let $\mathbb{U} = \mathbb{R}^L \oplus \mathbb{R}^L \oplus \mathbb{R}$, define

$$\mathcal{K} = \left\{ (\hat{\nu}, \hat{c}, t) \in \mathbb{R}^L \oplus \mathbb{R}^L \oplus \mathbb{R} : \hat{\nu}, \hat{c} \geq 0,\ \sum_{j=1}^k \hat{\nu}_j \log\left(\frac{\hat{\nu}_j}{\hat{c}_j}\right) \leq t \right\},$$

and define $T_n \colon \mathbb{U} \to \mathbb{U}_n^{\mathsf{G}_n}$ sending $(e_j, 0, 0) \mapsto |\mathsf{G}_n/\mathsf{Stab}_{\mathsf{G}_n}(\alpha_j)|^{-1}(\mathbb{1}_{j,n}, 0, 0)$, sending $(0, e_j, 0) \mapsto |\mathsf{G}_n/\mathsf{Stab}_{\mathsf{G}_n}(\alpha_j)|^{-1}(0, \mathbb{1}_{j,n}, 0)$, and sending $(0, 0, 1) \mapsto (0, 0, 1)$. Then $\mathrm{RE}_{\mathcal{A}_n}^{\mathsf{G}_n} = T_n(\mathcal{K})$ for all $n \geq d$, giving the desired constant-sized descriptions. ☐

Now suppose $\mathscr{W} = \{\mathbb{R}^{\mathcal{B}_n}\}$ is another permutation $\mathscr{V}_0$-module. Let $\widetilde{\mathscr{U}} = \mathscr{W} \otimes \mathscr{U} = \{\widetilde{\mathbb{U}_n} = \mathcal{L}(\mathbb{R}^{\mathcal{B}_n}, \mathbb{U}_n)\}$ and consider the cones of maps

$$\mathrm{REM}_{\mathcal{A}_n, \mathcal{B}_n} = \left\{ M \in \widetilde{\mathbb{U}_n} : M(\mathbb{R}_+^{\mathcal{B}_n}) \subseteq \mathrm{RE}_{\mathcal{A}_n} \right\} = \left\{ M \in \widetilde{\mathbb{U}_n} : Me_\beta \in \mathrm{RE}_{\mathcal{A}_n} \text{ for all } \beta \in \mathcal{B}_n \right\}. \tag{27}$$

We obtain constant-sized descriptions for these cones for a specific $\mathscr{V}_0$.

Proposition 5.5. *Suppose $\mathscr{V}_0 = \{\mathbb{R}^n\}$ with $\mathsf{G}_n = \mathsf{B}_n, \mathsf{D}_n$, or S_n as in Example 1.5 and that $\{\mathbb{R}^{\mathcal{A}_n}\}$, $\{\mathbb{R}^{\mathcal{B}_n}\}$ are permutation $\mathscr{V}_0$-modules generated in degrees d_V, d_W, respectively. Then the sequence of cones $\{\mathrm{REM}_{\mathcal{A}_n,\mathcal{B}_n}^{\mathsf{G}_n}\}$ in (27) admits constant-sized descriptions for $n \geq d_V + d_W$ if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ and $n \geq d_V + d_W + 1$ if $\mathsf{G}_n = \mathsf{D}_n$.*

Proof. Set $d_0 = 0$ if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ and $d_0 = 1$ if $\mathsf{G}_n = \mathsf{D}_n$. Let $\hat{\mathcal{B}} \subseteq \mathcal{B}_{d_W}$ be a set of minimal-degree $\mathsf{G}_{d_W+d_0}$-orbit representatives in $\mathcal{B}_{d_W+d_0}$, which are also orbit representatives for $\mathcal{B}_n$ for all $n \geq d_W + d_0$ by Proposition 2.15(c). Any $M \in \widetilde{\mathbb{U}_n}^{\mathsf{G}_n} = \mathcal{L}(\mathbb{R}^{\mathcal{B}_n}, \mathbb{U}_n)^{\mathsf{G}_n}$ for $n \geq d_W + d_0$ is fully determined by the images $Me_\beta \in \mathbb{U}_n^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)}$ of the basis elements e_β for $\beta \in \hat{\mathcal{B}}$ and conversely, for any collection $\left\{u_\beta \in \mathbb{U}_n^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)}\right\}_{\beta\in\hat{\mathcal{B}}}$ there is a unique $M \in \widetilde{\mathbb{U}_n}^{\mathsf{G}_n}$ satisfying $Me_\beta = u_\beta$. Moreover, $M \in \widetilde{\mathcal{K}_n}$ if and only if $Me_\beta \in \mathcal{K}_n^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)}$ for all $\beta \in \hat{\mathcal{B}}$. Thus, we have

$$\widetilde{\mathbb{U}_n}^{\mathsf{G}_n} = \bigoplus_{\beta\in\hat{\mathcal{B}}} \mathbb{U}_n^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)} \supseteq \bigoplus_{\beta\in\hat{\mathcal{B}}} \mathrm{RE}_{\mathcal{A}_n}^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)} = \mathrm{REM}_{\mathcal{A}_n,\mathcal{B}_n}^{\mathsf{G}_n}.$$

Applying Corollary 2.19 to the free module with which $\{\mathbb{R}^{\mathcal{A}_n}\}$ agrees for $n \geq d_V + d_0$ by Proposition 2.15(c), and which is presented in the same degree, we conclude that the projections $\mathcal{P}_{\mathbb{V}_n} \colon (\mathbb{R}^{\mathcal{A}_{n+1}})^{\mathsf{Stab}_{\mathsf{G}_{n+1}}(\beta)} \to (\mathbb{R}^{\mathcal{A}_n})^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)}$ are isomorphisms for all $n \geq d_V + d_W + d_0$. Proposition 5.4 then gives constant-sized descriptions for $\left\{\mathrm{RE}_{\mathcal{A}_n}^{\mathsf{Stab}_{\mathsf{G}_n}(\beta)}\right\}_n$ for each $\beta \in \hat{\mathcal{B}}$. ☐

As an application of Proposition 5.5, we obtain constant-sized descriptions for SAGE cones of signomials. Indeed, if $\mathcal{A}_n, \mathcal{B}_n \subseteq \mathbb{R}^n$ as in the above proposition, define the sequence $\mathscr{F} = \{\mathbb{F}_n\}$ of functions on $\{\mathbb{R}^n\}$

$$\mathbb{F}_n = \left\{ f(x) = \sum_{\alpha\in\mathcal{A}_n} c_\alpha e^{\langle\alpha,x\rangle} + \sum_{\beta\in\mathcal{B}_n} t_\beta e^{\langle\beta,x\rangle} : c_\alpha, t_\beta \in \mathbb{R} \right\} \cong \mathbb{R}^{\mathcal{A}_n} \oplus \mathbb{R}^{\mathcal{B}_n},$$

with G_n acting by $g \cdot f = f \circ g^{-1}$. Note that $\mathscr{F} = \mathscr{V} \oplus \mathscr{W}$ as consistent sequences, where $\mathscr{V} = \{\mathbb{R}^{\mathcal{A}_n}\}$ and $\mathscr{W} = \{\mathbb{R}^{\mathcal{B}_n}\}$ are as above. Sums of exponentials as in $\mathbb{F}_n$ are called signomials, and optimization problems involving such functions arise in a number of applications [30]. As usual, minimizing a signomial f over $\mathbb{R}^n$ can be recast as maximizing $\gamma \in \mathbb{R}$ such that $f - \gamma \geq 0$ on $\mathbb{R}^n$, so that optimizing signomials can be reduced to certifying their nonnegativity. This is NP-hard in general, but it can be done efficiently if only a single coefficient of f is nonnegative or if f is a sum of such signomials [29]. Formally, define the cones of (Sums of) nonnegative AM/GM Exponential functions, called AGE (resp., SAGE) functions, by

$$\mathrm{AGE}_{\mathcal{A}_n,\beta} = \left\{ f(x) = \sum_{\alpha\in\mathcal{A}_n} c_\alpha e^{\langle \alpha, x\rangle} + te^{\langle \beta, x\rangle} : f \geq 0 \text{ on } \mathbb{R}^n \text{ and } c_\alpha \geq 0 \text{ for all } \alpha \in \mathcal{A}_n \right\},$$
$$\mathrm{SAGE}_{\mathcal{A}_n,\mathcal{B}_n} = \sum_{\beta\in\mathcal{B}_n} \mathrm{AGE}_{\mathcal{A}_n,\beta}.$$

Theorem 5.6. *Suppose $\mathcal{A}_n, \mathcal{B}_n \subseteq \mathbb{R}^n$ where $\mathbb{R}^n$ is embedded in $\mathbb{R}^{n+1}$ by zero-padding and with the standard action of $\mathsf{G}_n = \mathsf{S}_n, \mathsf{D}_n$ or B_n. If $\mathcal{A}_n = \bigcup_{g\in\mathsf{G}_n} g\mathcal{A}_{d_A}$ for all $n \geq d_A$ and $\mathcal{B}_n = \bigcup_{g\in\mathsf{G}_n} g\mathcal{A}_{d_B}$ for all $n \geq d_B$, then the invariant SAGE cones $\{\mathrm{SAGE}^{\mathsf{G}_n}_{\mathcal{A}_n,\mathcal{B}_n}\}_n$ admit constant-sized descriptions for $n \geq d_A + d_B$ if $\mathsf{G}_n = \mathsf{S}_n, \mathsf{B}_n$ and $n \geq d_A + d_B + 1$ if $\mathsf{G}_n = \mathsf{D}_n$.*

Proof. Identify $M \in \widetilde{\mathbb{U}_n}$ with tuples $(Me_\beta)_{\beta\in\mathcal{B}_n} = (\nu^{(\beta)}, c^{(\beta)}, t_\beta)_{\beta\in\mathcal{B}_n} \in \mathbb{R}^{\mathcal{A}_n} \oplus \mathbb{R}^{\mathcal{A}_n} \oplus \mathbb{R}$ for each $\beta \in \mathcal{B}_n$. The authors of [29] show that, in our notation,

$$\begin{aligned}\mathrm{SAGE}_{\mathcal{A}_n,\mathcal{B}_n} &= \Big\{ (c,t) \in \mathbb{R}^{\mathcal{A}_n} \oplus \mathbb{R}^{\mathcal{B}_n} : \exists M = (\nu^{(\beta)}, c^{(\beta)}, t_\beta)_{\beta\in\mathcal{B}_n} \in \mathrm{REM}_{\mathcal{A}_n,\mathcal{B}_n} \text{ s.t. } \textstyle\sum_{\beta\in\mathcal{B}_n} c^{(\beta)} = c, \\ &\qquad \textstyle\sum_{\alpha\in\mathcal{A}_n} \nu^{(\beta)}_\alpha (\alpha - \beta) = 0 \text{ for all } \beta \in \mathcal{B}_n \Big\} \\ &= \pi_n(\mathrm{REM}_{\mathcal{A}_n,\mathcal{B}_n} \cap \mathbb{L}_n),\end{aligned}$$

where

$$\mathbb{L}_n = \Big\{ M = (\nu^{(\beta)}, c^{(\beta)}, t_\beta)_{\beta\in\mathcal{B}_n} \in \widetilde{\mathbb{U}_n} : \textstyle\sum_{\alpha\in\mathcal{A}_n} \nu^{(\beta)}_\alpha (\alpha - \beta) = 0 \text{ for all } \beta \in \mathcal{B}_n \Big\},$$
$$\pi_n(M) = \Big(\textstyle\sum_{\beta\in\mathcal{B}_n} c^{(\beta)}, (t_\beta)_{\beta\in\mathcal{B}_n} \Big) \in \mathbb{R}^{\mathcal{A}_n} \oplus \mathbb{R}^{\mathcal{B}_n}.$$

Note that π_n is equivariant, since

$$(g \cdot M)e_\beta = gMg^{-1}e_\beta = gMe_{g^{-1}\beta} = g \cdot (\nu^{(g^{-1}\beta)}, c^{(g^{-1}\beta)}, t_{g^{-1}\beta}) = (g \cdot \nu^{(g^{-1}\beta)}, g \cdot c^{(g^{-1}\beta)}, t_{g^{-1}\beta}),$$

hence

$$\pi_n(g \cdot M) = \left(\sum_{\beta\in\mathcal{B}_n} g \cdot c^{(g^{-1}\beta)}, (t_{g^{-1}\beta})_{\beta\in\mathcal{B}_n} \right) = \left(g \cdot \sum_{\beta\in\mathcal{B}_n} c^{(\beta)}, g \cdot (t_\beta)_{\beta\in\mathcal{B}_n} \right) = g \cdot \pi_n(M).$$

Similarly, if $\sum_{\alpha\in\mathcal{A}_n} \nu^{(\beta)}_\alpha (\alpha - \beta) = 0$ for all $\beta \in \mathcal{B}_n$ then

$$\sum_{\alpha\in\mathcal{A}_n} (g \cdot \nu^{(g^{-1}\beta)})_\alpha (\alpha - \beta) = g \sum_{\alpha\in\mathcal{A}_n} \nu^{(g^{-1}\beta)}_{g^{-1}\alpha} (g^{-1}\alpha - g^{-1}\beta) = g \sum_{\alpha\in\mathcal{A}_n} \nu^{(g^{-1}\beta)}_\alpha (\alpha - g^{-1}\beta) = 0,$$

hence $\mathbb{L}_n$ is G_n-invariant. Thus, the result follows from Proposition 5.5 and Proposition 5.3. ☐

Theorem 5.6 generalizes [27, Thm. 5.3] beyond S_n to the other classical Weyl groups D_n and B_n. It would be interesting to further generalize to signomials defined on more general consistent sequences than $\{\mathbb{R}^n\}$, which would require generalizing the description of the AGE cone from [29].

6 Free Convex Regression

In this section, we use our framework to describe a solution to the free convex regression problem. Recall that in this problem we are given evaluation data in different dimensions and we seek a sequence of convex functions that best fit the data and can be instantiated in any desired dimension (including those not represented in the data). To that end, we use the framework we developed in Section 3 to obtain parametric families of freely-described convex sets in a fully algorithmic manner. We then present a numerical procedure to fit elements of these families to the given data. Our implementation of the resulting algorithms can be found at `https://github.com/eitangl/anyDimCvxSets`.

6.1 Computationally Parametrizing Descriptions

Suppose we seek a parametric family of convex subsets $\{\mathcal{C}_n \subseteq \mathbb{V}_n\}$ of a consistent sequence $\mathscr{V} = \{\mathbb{V}_n\}$ of $\mathscr{G} = \{\mathsf{G}_n\}$-representations. Both $\mathscr{V}$ and $\mathscr{G}$ are usually dictated by the application at hand and the symmetries it exhibits, as in Example 1.7. We then select description spaces $\mathscr{U} = \{\mathbb{U}_n\}, \mathscr{W} = \{\mathbb{W}_n\}$, usually constructed from $\mathscr{V}$ as described in Section 2.3. Their choice is dictated by the desired richness of the family of the freely-described sets and, as we shall see in Remark 6.3 below, by the dimensions of the available data. Once all of these are chosen, parametrizing the associated family of freely-described convex sets amounts to identifying bases for the spaces of freely-described vectors and linear maps appearing in (ConicSeq). Up to this point we have obtained such bases analytically, as the relevant spaces of invariants have been fairly low-dimensional, but this approach becomes impractical for richer description spaces. To address this challenge, we present a computational method for obtaining the relevant bases, which yields a fully algorithmic approach for deriving parametrized families of freely-described convex sets.

Computing a basis for the space of freely-described elements in a finitely-generated consistent sequence proceeds in two steps. First, we compute a basis for the space of invariants in a fixed, sufficiently large dimension. Second, we extend the elements of this basis to any other dimension, which is done by solving a linear system. Our procedure is summarized in Algorithm 1, which we proceed to describe in more detail.

Algorithm 1 Computationally parametrize a freely-described (and possibly compatible) sequence of convex sets.

1: **Input:** Consistent sequences $\mathscr{V}, \mathscr{W}, \mathscr{U}$.
2: **Output:** Bases $\{\mathscr{A}^{(i)} = \{A_n^{(i)}\}_{n\in\mathbb{N}}\}_{i=1}^{d_A}, \{\mathscr{B}^{(j)} = \{B_n^{(j)}\}_{n\in\mathbb{N}}\}_{j=1}^{d_B}$ for freely-described equivariant maps or morphisms.
3: **if** compatibility not required **then**
4:     Fix $n_0 \geq$ presentation degrees of $\mathscr{V} \otimes \mathscr{U}$ and $\mathscr{W} \otimes \mathscr{U}$.
5:     Find bases $\{A_{n_0}^{(i)}\}_{i=1}^{d_A}, \{B_{n_0}^{(j)}\}_{j=1}^{d_B}$ for $\mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ and $\mathcal{L}(\mathbb{W}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$.
6: **else**
7:     Fix $n_0 \geq$ presentation degrees of $\mathscr{V}, \mathscr{W}, \mathscr{U}$.
8:     Find bases $\{A_{n_0}^{(i)}\}_{i=1}^{d_A}, \{B_{n_0}^{(j)}\}_{j=1}^{d_B}$ for subspaces of $\mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ and $\mathcal{L}(\mathbb{W}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ satisfying the hypotheses of Theorem 3.5.
9: **end if**
10: For any $n > n_0$ and each i, j, find unique equivariant $A_n^{(i)}, B_n^{(j)}$ projecting onto $A_{n_0}^{(i)}, B_{n_0}^{(j)}$. For each $n < n_0$, project $A_{n_0}^{(i)}, B_{n_0}^{(j)}$ to dimension n to obtain $A_n^{(i)}, B_n^{(j)}$.

Fix $n_0 \in \mathbb{N}$ as in Algorithm 1. We now explain how to find bases for the desired spaces of equivariant linear maps A_{n_0} and B_{n_0} in a fixed dimension, and how to extend them to bases of freely-described elements of $\mathscr{V} \otimes \mathscr{U}$ and $\mathscr{W} \otimes \mathscr{U}$. These elaborate on steps 5, 8 and 10 in Algorithm 1.

Step 5: Computing basis for equivariant maps. We explain how to compute a basis for invariants in a fixed vector space, which we then instantiate in the context of Algorithm 1 to perform step 5. If $\mathbb{V}$ is a representation of a group G, a vector $v \in \mathbb{V}$ is G-invariant iff $g \cdot v = v$ for all $g \in \mathsf{G}$, which can be rewritten as $v \in \ker(g - I)$ for all $g \in \mathsf{G}$. Thus, finding a basis for invariants in a fixed vector space reduces to finding a basis for the kernel of a matrix, though this matrix may be very large or even infinite. We can dramatically reduce the size of this matrix by only considering discrete and continuous generators of G, as proposed in [31].

Theorem 6.1 ([31, Thm. 1])**.** *Let G be a real Lie group with finitely-many connected components acting on a vector space $\mathbb{V}$ via the homomorphism $\rho\colon \mathsf{G} \to \mathrm{GL}(\mathbb{V})$. Let $\{H_i\}$ be a basis for the Lie algebra $\mathfrak{g}$ of G and $\{h_j\}$ be a finite collection of discrete generators. Then*

$$v \in \mathbb{V}^{\mathsf{G}} \iff \mathrm{D}\rho(H_i)v = 0 \text{ and } (\rho(h_j) - \mathrm{id}_{\mathbb{V}}) \cdot v = 0 \quad \text{for all } i, j.$$

Here $\mathrm{D}\rho\colon \mathfrak{g} \to \mathcal{L}(\mathbb{V})$ is the differential of ρ. Sets of Lie algebra bases and discrete generators for various standard groups are given in [31]. For example, $\mathsf{G} = \mathsf{S}_n$ is generated by two elements, namely, the transposition $(1, 2)$ and the n-cycle $(1, \ldots, n)$, reducing the number of group elements that must be considered from the naïve $n!$ to two. For $\mathsf{G} = \mathsf{O}_n$, a basis for the Lie algebra $\mathfrak{g} = \mathrm{Skew}(n)$ is given by $E_{i,j} = e_i e_j^\top - e_j e_i^\top$ for $i < j$, and only one discrete generator, e.g., $h_1 = \mathrm{diag}(-1, 1, \ldots, 1)$, is needed, for a total of $\binom{n}{2} + 1$ elements.

As equivariant linear maps are precisely the invariants in the space of linear maps, Theorem 6.1 allows us to obtain a basis for equivariant maps between fixed vector spaces. Explicitly, if $\rho_V : \mathsf{G}_{n_0} \to \mathrm{GL}(\mathbb{V}_{n_0})$ and $\rho_U : \mathsf{G}_{n_0} \to \mathrm{GL}(\mathbb{U}_{n_0})$ are the group homomorphisms defining the actions of G_{n_0} on $\mathbb{V}_{n_0}, \mathbb{U}_{n_0}$, then $A_{n_0} \in \mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ if and only if

$$\mathrm{D}\rho_U(H_i)A_{n_0} - A_{n_0}\mathrm{D}\rho_V(H_i) = 0, \quad \rho_U(h_j)A_{n_0} - A_{n_0}\rho_V(h_j) = 0, \quad \text{for all } i, j. \tag{28}$$

The equations (28) express the space $\mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ as the kernel of a matrix, which is often very large and sparse. A basis for the kernel of such a matrix can be computed using its LU decomposition as in [84, 85], or using the algorithm of [31, §5]. A basis for the space $\mathcal{L}(\mathbb{W}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ is obtained analogously.

Step 8: Computing a basis for extendable equivariant linear maps. We know from Example 3.1 that extending arbitrary invariants $A_{n_0}, B_{n_0}, u_{n_0}$ to freely-described elements does not yield a compatible sequence of convex sets. However, Theorem 3.5 identifies subspaces of invariant linear maps A_{n_0}, B_{n_0} whose extensions do yield a compatible sequence of sets, provided we fix a freely-described element $\{u_n\}$ satisfying $u_{n+1} - u_n \in \mathcal{K}_n$. Specifically, we need to find a basis for equivariant linear maps satisfying $A_{n_0}(\mathbb{V}_j) \subseteq \mathbb{U}_j$ for $j \leq d_V$ where d_V is the generation degree of $\mathscr{V}$. This is again a linear condition on A_{n_0}; defining $\varphi_{n_0,j} = \varphi_{n_0-1} \cdots \varphi_j : \mathbb{V}_j \hookrightarrow \mathbb{V}_{n_0}$ if $j < n_0$ and $\varphi_{n_0,n_0} = \mathrm{id}_{\mathbb{V}_{n_0}}$, and similarly for $\psi_{n_0,j}$, we have

$$A_{n_0}(\mathbb{V}_j) \subseteq \mathbb{U}_j \iff (I - \mathcal{P}_{\mathbb{U}_j})A_{n_0}|_{\mathbb{V}_j} = 0 \iff (I - \psi_{n_0,j}\psi_{n_0,j}^*)A_{n_0}\varphi_{n_0,j} = 0. \tag{29}$$

The subspace of $\mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ satisfying the hypotheses of Theorem 3.5 is thus again the kernel of a matrix obtained by combining (28) and (29). To also impose $A_{n_0}(\mathbb{V}_j^\perp) \subseteq \mathbb{U}_j^\perp$ for $j \leq d_U$ where d_U is the generation degree of $\mathscr{U}$, so that $A_{n_0}^*$ extends to a morphism, note that

$$A_{n_0}(\mathbb{V}_j^\perp) \subseteq \mathbb{U}_j^\perp \iff \mathcal{P}_{\mathbb{U}_j} A_{n_0}|_{V_i^\perp} = 0 \iff \psi_{n_0,i}^* A_{n_0}(I - \varphi_{n_0,i}\varphi_{n_0,i}^*) = 0, \tag{30}$$

hence the corresponding subspace of $\mathcal{L}(\mathbb{V}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ is the kernel of the matrix obtained by combining (28)-(30). The subspace of $\mathcal{L}(\mathbb{W}_{n_0}, \mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ satisfying the hypotheses of Theorem 3.5 is again the kernel of a matrix and its basis is computed similarly.

Step 10: Extending bases to higher dimensions. Given bases of equivariant $A_{n_0}^{(i)}, B_{n_0}^{(j)}$, we wish to extend them to freely-described elements. We do so computationally by applying a linear map for $n < n_0$ and solving a linear system for each $n > n_0$ to which we wish to extend. For each $n < n_0$, we set $A_n^{(i)} = \psi_{n_0,n}^* A_{n_0}^{(i)} \varphi_{n_0,n}$ and similarly for $B_n^{(j)}$. For each $n > n_0$, we set $A_n^{(i)}$ to be the unique solution to the linear system (28) (with n_0 replaced by n) and $\psi_{n,n_0}^* A_n^{(i)} \varphi_{n,n_0} = A_{n_0}^{(i)}$. This linear system is typically large and sparse, and we solve it using LSQR [86]. The extension of $B_{n_0}^{(j)}$ is handled similarly, except that n_0 needs to exceed the presentation degrees of both $\mathscr{W}$ and $\mathscr{U}$ to guarantee that both $B_{n_0}^{(j)}$ and $(B_{n_0}^{(j)})^*$ extend to morphisms.

Example 6.2 (Dimension counts)**.** *We use the above algorithm to obtain dimension counts for parametric families of free descriptions. See the functions `compute_dims_a`, `compute_dims_b`, and `compute_dims_c` on GitHub for the code computing these dimensions using Algorithm 1.*

(a) *Let* $\mathscr{V} = \{\mathbb{R}^n\}$ *with* $\mathsf{G}_n = \mathsf{S}_n$ *as in Example 1.5, and let* $\mathscr{W} = \mathscr{U} = \mathrm{Sym}^2(\mathrm{Sym}^{\leq 2}\mathscr{V})$*. Then* $\mathscr{V}, \mathscr{U}, \mathscr{V} \otimes \mathscr{U}, \mathscr{W} \otimes \mathscr{U}$ *are all* $\mathscr{V}$*-modules and are presented in degrees* $1, 4, 5, 8$*, respectively, by Theorem 2.11.*

The dimensions of invariants parametrizing free descriptions are $\dim \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n} = 39$, $\dim \mathcal{L}(\mathbb{W}_n, \mathbb{U}_n)^{\mathsf{G}_n} = 1068$*, and* $\dim \mathbb{U}_n^{\mathsf{G}_n} = 17$ *for* $n \geq 8$*. The dimensions of linear maps* $\{A_n\}$ *and* $\{B_n\}$ *satisfying Proposition 3.2(a), yielding intersection-compatible convex sets, are*

$$\dim \Big\{\{A_n \colon \mathbb{V}_n \to \mathbb{U}_n\} \textit{ morphism}\Big\} = 6,$$
$$\dim \Big\{\{B_n \colon \mathbb{W}_n \to \mathbb{U}_n\} : \textit{both } \{B_n\} \textit{ and } \{B_n^*\} \textit{ morphisms}\Big\} = 104.$$

If we further require $\{A_n^*\}$ *to be a morphism to obtain projection compatibility, the dimension of* $\{A_n\}$ *decreases to 5.*

(b) *Let* $\mathscr{V} = \{\mathbb{S}^n\}$ *with* $\mathsf{G}_n = \mathsf{S}_n$ *used to obtain graph parameters in Example 1.7, and let* $\mathscr{W}, \mathscr{U}$ *be as in (a). Then* $\mathscr{V}, \mathscr{U}, \mathscr{V} \otimes \mathscr{U}, \mathscr{W} \otimes \mathscr{U}$ *are all* $\mathscr{V}$*-modules and are presented in degrees* $2, 4, 6, 8$*, respectively.*

The dimension of invariant $\{A_n\}$ *in this case is* $\dim \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n} = 93$*, and the dimensions of* $\dim \mathcal{L}(\mathbb{W}_n, \mathbb{U}_n)^{\mathsf{G}_n}$ *and* $\dim \mathbb{W}_n^{\mathsf{G}_n}$ *are as in (a), giving the number of parameters describing freely-described convex sets with these description spaces. The dimensions of linear maps* $\{A_n\}$ *and* $\{B_n\}$ *satisfying Proposition 3.2(a), and hence describing intersection-compatible sets, are 19 and 104, respectively, as given in (18). If we further require* $\{A_n^*\}$ *to be a morphism to get projection compatibility, the dimension of* $\{A_n\}$ *decreases to 12.*

(c) *Let* $\mathscr{V} = \{\mathbb{R}^n\}$ *with* $\mathsf{G}_n = \mathsf{B}_n$ *as in Example 1.5, used below to learn regularizers defined for vectors of any length, let* $\mathscr{V}' = \{\mathbb{R}^{2n+1}\} = \mathscr{W}^{(2)} \oplus \mathbb{R}$ *as in Example 3.1, and let* $\mathscr{W} = \mathscr{U} = \mathrm{Sym}^2(\mathrm{Sym}^{\leq 1}\mathscr{V}')$*. Then* $\mathscr{V}, \mathscr{U}, \mathscr{V} \otimes \mathscr{U}, \mathscr{W} \otimes \mathscr{U}$ *are all* $\mathscr{V}$*-modules and are presented in degrees* $1, 2, 3, 4$*, respectively.*

The dimensions of invariants parametrizing freely-described convex sets in this case are

$$\dim \mathcal{L}(\mathbb{V}_n, \mathbb{U}_n)^{\mathsf{G}_n} = 4, \quad \dim \mathcal{L}(\mathbb{W}_n, \mathbb{U}_n)^{\mathsf{G}_n} = 108, \quad \dim \mathbb{U}_n^{\mathsf{G}_n} = 8, \quad \textit{for all } n \geq 4.$$

The dimensions of linear maps $\{A_n\}$ *and* $\{B_n\}$ *satisfying Proposition 3.2(a) and parametrizing intersection-compatible sets are*

$$\begin{aligned} &\dim \Big\{\{A_n \colon \mathbb{V}_n \to \mathbb{U}_n\} \textit{ morphism}\Big\} = 3, \\ &\dim \Big\{\{B_n \colon \mathbb{W}_n \to \mathbb{U}_n\} : \textit{both } \{B_n\} \textit{ and } \{B_n^*\} \textit{ morphisms}\Big\} = 37. \end{aligned} \tag{31}$$

If we further require $\{A_n^*\}$ *to be a morphism, the dimension does not decrease in this case, so all of these intersection-compatible sets are also projection-compatible.*

6.2 Fitting Freely-Described Convex Functions to Data

We now present an algorithm for the free convex regression problem from Section 1.1.4. Recall that in this problem we are given data $\{(x_i, \phi_i) \in \mathbb{V}_{n_i} \oplus \mathbb{R}\}$ in finitely-many dimensions n_i corresponding to a sequence of vector spaces $\{\mathbb{V}_n\}$, and our objective is to identify a sequence of convex functions $\{f_n : \mathbb{V}_n \to \mathbb{R}\}$ such that $f_{n_i}(x_i) \approx \phi_i$ in the dimensions in which data is available. We tackle this problem by endowing $\{\mathbb{V}_n\}$ with the structure of a consistent sequence of $\{\mathsf{G}_n\}$-representations, and choosing description spaces $\{\mathbb{W}_n\}, \{\mathbb{U}_n\}$ that yield a finitely-parametrized family of freely-described and compatible convex functions which we fit to the given data. We explain how to perform this fitting below in two key steps.

The first step is to define a finitely-parametrized family of freely-described convex functions. Let $\{\{A_n^{(i)}\}\}_{i=1}^{d_A}$ and $\{\{B_n^{(j)}\}\}_{j=1}^{d_B}$ be the bases for freely-described maps computed by Algorithm 1 where we choose $n_0 \geq \max_i n_i$ so that we do not have to extend the basis elements computed there to access the data dimensions. Further, we select intersection and projection compatible cones $\mathscr{K} = \{\mathcal{K}_n \subseteq \mathbb{U}_n\}$, and freely-described $\{u_n \in \mathbb{U}_n^{\mathsf{G}_n}\}$ satisfying $u_{n+1} - u_n \in \mathcal{K}_n$ and $u_n \in \mathrm{int}(\mathcal{K}_n)$ for all n. We consider families of nonnegative and positively-homogeneous convex functions f_n parametrized by $\alpha \in \mathbb{R}^{d_A}$, $\beta \in \mathbb{R}^{d_B}$, and $\lambda \in \mathbb{R}$ of the form

$$f_n(x;\alpha,\beta,\lambda) = \inf_{\substack{t\geq 0\\ y\in\mathbb{W}_n}} t+\lambda\|y\| \quad \text{s.t.} \ \sum_{i=1}^{d_A}\alpha_i A_n^{(i)}x + \sum_{j=1}^{d_B}\beta_j B_n^{(j)}y + tu_n \in \mathcal{K}_n \tag{P}$$

$$= \sup_{z\in\mathbb{U}_n} -\sum_{i=1}^{d_A}\alpha_i\langle z, A_n^{(i)}x\rangle \quad \text{s.t.} \ \left\|\sum_{j=1}^{d_B}\beta_j(B_n^{(j)})^*z\right\| \leq \lambda,\ \langle z,u_n\rangle\leq 1,\ z\in\mathcal{K}_n^*, \tag{D}$$

where the primal and dual programs above are equal since Slater's condition is satisfied by our choice of u_n. Here $\|\cdot\|$ is the norm on $\mathbb{W}_n$ induced by its inner-product and the parameter λ is chosen to be positive; the purpose of this term in (P) is to prevent numerical issues arising in our subsequent regression procedure (see (Regress)). Note that this is indeed a finitely-parametrized (by $d_A + d_B + 1$ parameters) infinite sequence of functions. Also note that if we require compatibility in Algorithm 1, then the sequence $\{f_n\}$ is intersection (and if desired, projection) compatible for any value of the parameters, since it is the sequence of gauge functions of a correspondingly compatible sequence of sets (see Section 1.3).

The second step concerns optimizing over the parameters α, β, λ to fit $\{f_n\}$ to the available data. To this end, we consider the following optimization problem, with a user-specified $\lambda_{\min} > 0$:

$$\min_{\substack{\varepsilon\in\mathbb{R}^D_+\\ \alpha\in\mathbb{R}^{d_A},\beta\in\mathbb{R}^{d_B},\lambda\geq\lambda_{\min}\\ \{(t_i,y_i)\},\{z_i\}}} \|\varepsilon\|_{\ell_2} \quad \text{s.t.} \tag{Regress}$$

$$(y_i,t_i) \text{ feasible for (P) with } n=n_i \text{ and cost } \leq \phi_i+\varepsilon_i, \tag{PC}$$

$$z_i \text{ feasible for (D) with } n=n_i \text{ and cost } \geq \phi_i-\varepsilon_i,. \tag{DC}$$

The constraints (PC) and (DC) are required to hold for all $i \in [D]$ and they ensure that $\phi_i - \varepsilon_i \leq f_{n_i}(x_i;\alpha,\beta,\lambda) \leq \phi_i + \varepsilon_i$, so that minimizing $\|\varepsilon\|_{\ell_2}$ yields a good fit to the data. We emphasize that this problem is finite-dimensional even though the it yields an infinite sequence of convex functions $\{f_n\}$.

As (Regress) involves bilinear constraints, we solve the problem via *alternating minimization*, where we alternate between fixing α,β,λ and $\{(t_i,y_i)\},\{z_i\}$ while optimizing over the rest of the variables. Note that Slater's condition holds in (Regress) for both steps of alternating minimization when $u_n \in \mathrm{int}(\mathcal{K}_n)$.

Remark 6.3. *The quantity n_0 in Algorithm 1 is governed by the choice of description spaces and leads to a tradeoff between the richness of the parametric family and the dimensions of the available data. The data we are given only contains information about $\{f_n\}_{n\leq\max_i n_i}$ which, in turn, only depends on $\{\{A_n^{(i)}\}_{n\leq\max_i n_i}\}_{i=1}^{d_A}$ and $\{\{B_n^{(j)}\}_{n\leq\max_i n_i}\}_{j=1}^{d_B}$. If n_0 in Algorithm 1 is strictly larger than the maximum dimension $\max_i n_i$ in which data is available, then the number of distinct basis elements for $n\leq\max_i n_i$ might be strictly smaller than d_A and d_B, respectively, in which case our parameters α,β are not identifiable from such low-dimensional data. Imposing compatibility decreases the bound on n_0, thus facilitating free convex regression from lower-dimensional data. More broadly, even when $\max_i n_i \geq n_0$, it is of interest to investigate the landscape of* (Regress).

6.3 Numerical Results

In all experiments below, we use $\mathscr{U} = \mathrm{Sym}^2(\mathrm{Sym}^{\leq k}\mathscr{V}')$ for some $\mathscr{V}$-module $\mathscr{V}'$ and some k, with the corresponding PSD cones $\mathscr{K} = \mathrm{Sym}^2_+(\mathrm{Sym}^{\leq k}\mathscr{V}')$. We choose $\{u_n\}$ to be the sequence of identity matrices, which satisfy $u_n \in \mathrm{int}(\mathcal{K}_n)$ and $u_{n+1} - u_n \in \mathcal{K}_n$ as required. We apply our algorithm to learn semidefinite approximations of two non-SDP-representale functions, comparing the results obtained with and without imposing compatibility in Algorithm 1.

The first function we approximate is the ℓ_π norm $\|x\|_\pi = (\sum_i |x_i|^\pi)^{1/\pi}$, which is not SDP-representable because π is irrational. We view the ℓ_π norm as defined on the sequence $\mathscr{V} = \{\mathbb{R}^n\}$ with $\mathsf{G}_n = \mathsf{B}_n$ from Example 1.5. It satisfies both intersection and projection compatibility. We choose description spaces $\mathscr{W} = \mathscr{U} = \{\mathrm{Sym}^2(\mathrm{Sym}^{\leq 1}\mathbb{R}^{2n+1}) = \mathbb{S}^{2n+2}\}$ as in Example 6.2(c), with the corresponding PSD cones $\{\mathcal{K}_n =$

$\mathbb{S}_+^{2n+2}\}$. We used 50 data points in $\mathbb{R}^n$ for $n_i \in \{1,2\}$. When we impose compatibility on our family of functions, we use $n_0 = 2 = \max_i n_i$ in Algorithm 1. If we do not impose compatibility and search over all freely-described (possibly incompatible) sequences, we take $n_0 = 4$ which is the presentation degree of $\mathscr{W} \otimes \mathscr{U}$, and which strictly exceeds $\max_i n_i$. The constraints in the fitting program (Regress) do not depend on some of the entries of α, β in this case, and we set such entries to zero. This highlights the advantage of imposing compatibility mentioned in Remark 6.3–it allows us to uniquely identify a free description from lower-dimensional data.

The second sequence of functions we approximate is the nonnegative and positively-homogeneous variant of the quantum entropy given by (2) given in Section 1.1.4, defined on the sequence $\{\mathbb{S}^n\}$ with embeddings by zero-padding and the action of $\mathsf{G}_n = \mathsf{O}_n$ by conjugation. Once again, the function (2) cannot be evaluated using semidefinite programming, though a family of semidefinite approximations is analytically derived in [36]. Here we aim to learn a semidefinite approximation entirely from evaluation data. To that end, we choose description spaces $\mathscr{W} = \{\mathbb{W}_n = \mathrm{Sym}^2(\mathrm{Sym}^{\leq 1}\mathbb{R}^n) = \mathbb{S}^{n+1}\}$ and $\mathscr{U} = \{\mathbb{U}_n = \mathrm{Sym}^2(\mathrm{Sym}^{\leq 2}\mathbb{R}^n) = \mathbb{S}^{\binom{n+2}{2}}\}$, with corresponding PSD cones $\{\mathcal{K}_n = \mathbb{S}_+^{\binom{n+2}{2}}\}$. Our data consists of 200 PSD matrices in $\mathbb{S}^n$ for $n \leq n_0 = 4$. Without a calculus for presentation degrees for $\mathsf{G}_n = \mathsf{O}_n$, our theory does not guarantee the existence of an O_n-invariant extension of our learned description. Our theory does however guarantee a unique B_n-invariant extension, and in practice we observe that the extension is, in fact, O_n-invariant.

To approximate the above functions, which we generically denote $\{f_n^{\mathrm{true}}\}$, we used (Regress) with 100 random initializations to fit the data in degree n_0. For the above two examples, not only is $f_{n_0}^{\mathrm{true}}$ positively-homogeneous and nonnegative, but also $f_{n_0}^{\mathrm{true}}(x) \neq 0$ for $x \neq 0$ in the domain. We therefore normalize the data x_i by $x_i \mapsto x_i / f_{n_0}^{\mathrm{true}}(x_i)$, so that $f_{n_0}^{\mathrm{true}}(x_i) = 1$ for all i and all points contribute equally to the objective of (Regress). We impose $\lambda \geq \lambda_{\min} = 10^{-3}$ in (Regress). To evaluate the results, we extended our learned descriptions to $n = 20$, sampled 10^3 unit-norm points (also PSD for the quantum entropy example) and computed the average normalized errors $\frac{|f_n(x) - f_n^{\mathrm{true}}(x)|}{f_n^{\mathrm{true}}(x)}$ in each n up to 20.

The resulting errors are plotted in Figure 1, shown and discussed in Section 1.1.4. Since imposing compatibility conditions decreases the search space in (Regress), we expect the optimal solution of (Regress) with compatibility conditions to exhibit larger errors in dimensions in which data is available compared to the optimal freely-described (but possibly incompatible) solution. That is not the case in Figure 1(b), illustrating the nonconvexity of the fitting problem (Regress) and demonstrating another advantage of imposing compatibility—the resulting smaller parametric family allows our algorithm to better fit the data.

7 Conclusions and Future work

We developed a systematic framework to study convex sets that can be instantiated in different dimensions using representation stability, as well as a computational method to parametrize such sets and fit them to data. We did so by formally defining free descriptions of convex sets and compatibility conditions relating sets in different dimensions. We then proved a number of structural results pertaining to free descriptions, namely, characterizing such descriptions certifying compatibility; giving conditions on fixed-dimensional descriptions ensuring their extendability to free ones; and studying infinite-dimensional limits of freely-described sequences of sets. We further used representation stability to systematically derive constant-sized descriptions for sequences of invariant sections of PSD and relative entropy cones. Finally, we developed an algorithm to computationally parametrize and search over free descriptions to fit them to data. Our work can be viewed as identifying and exploiting a new point of contact between representation stability and convex geometry through conic descriptions of convex sets.

Our work suggests questions and directions for future research in several areas.

(Computational algebra) Is there an algorithm to compute the generation and presentation degrees of a given consistent sequence?

(Lie groups) Can we extend our calculus for presentation degrees in Theorem 2.11 to Lie groups such as $\mathsf{G}_n = \mathsf{O}_n$?

(Constructing descriptions) Given a sequence of convex sets instantiable in any relevant dimension, can we systematically construct freely-described, and possibly compatible, approximations for the sequence? When are approximations derived from sums-of-squares hierarchies such as [38] free and certify compatibility?

(Complexity) Is there a systematic framework to study the smallest possible size of a free description for a given sequence of sets, extending the slack operator-based approach for fixed convex sets [87]?

(Free separation) Under what conditions can a point outside a compatible sequence of convex sets be separated by a freely-described sequence, generalizing the Effros-Winkler theorem [88]?

(Statistical inference) How much data do we need to learn a given sequence of sets or functions, and in what dimensions should this data lie?

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A Representations of Categories

As Remark 2.10 shows, the set of embeddings from low to high dimensions in a consistent sequence $\mathscr{V}$ of $\{\mathsf{G}_n\}$-representations, determined by $\{\mathsf{G}_n\}$ and the centralizing subgroups $\{\mathsf{H}_{n,d}\}$ from Definition 2.4, play a central role in our framework. These sets of embeddings are conveniently encoded in a category, whose representations are precisely the $\mathscr{V}$-modules of Definition 2.5. Morphisms between such representations in the categorical sense coincide with morphisms of sequences. This categorical approach to representation stability was introduced in [49] for the case $\mathsf{G}_n = \mathsf{S}_n$ and the $\mathsf{H}_{n,d}$ from Example 2.8, and has since been extended to other groups in [50, 51, 54].

Definition A.1. *A (real)* representation *of a category $\mathcal{C}$, also called a $\mathcal{C}$-*module*, is a functor $\mathcal{C} \to \mathrm{Vect}_{\mathbb{R}}$ from $\mathcal{C}$ to the category of real vector spaces.*

In other words, a $\mathcal{C}$-module is an assignment of a vector space $\mathbb{V}_n$ to each object $n \in \mathcal{C}$ and a linear map $\varphi_{n,N}\colon \mathbb{V}_n \to \mathbb{V}_n$ to each morphism in $\mathrm{Hom}_{\mathcal{C}}(n,N)$ such that compositions are respected. Each $\mathbb{V}_n$ is then a representation of the group $\mathsf{G}_n = \mathrm{End}_{\mathcal{C}}(n)^{\times}$ of the automorphisms of n in $\mathcal{C}$. Every consistent sequence is a representation of a suitable category.

Definition A.2. *Given a consistent sequence $\mathscr{V} = \{(\mathbb{V}_n, \varphi_n)\}$ of $\{\mathsf{G}_n\}$-representations, define a category $\mathcal{C}_{\mathscr{V}}$ whose set of objects is $\mathbb{N}$ and whose morphisms are $\mathrm{Hom}_{\mathcal{C}_V}(n,N) = \{g\varphi_{N-1}\cdots\varphi_n : g \in \mathsf{G}_N\}$ for $n \le N$ and zero otherwise. Note that $\mathrm{Hom}_{\mathcal{C}_V}(n,N) = \mathsf{G}_N/\mathsf{H}_{N,n}$ where $\mathsf{H}_{N,n} \subseteq \mathsf{G}_N$ is the subgroup of elements acting trivially on $\mathbb{V}_n$.*

This definition clearly extends to consistent sequences indexed by posets (Remark 4.4). If $\mathscr{U} = \{(\mathbb{U}_n, \psi_n)\}$ is a $\mathscr{V}$-module (Definition 2.5), then $\mathscr{U}$ is a $\mathcal{C}_{\mathscr{V}}$-module, since sending $n \in \mathbb{N}$ to $\mathbb{U}_n$ and $g\varphi_{N-1}\cdots\varphi_n$ to the map $g\psi_{N-1}\cdots\psi_n$ for each $g \in \mathsf{G}_N$ is a well-defined functor $\mathcal{C}_{\mathscr{V}} \to \mathrm{Vect}_{\mathbb{R}}$. Conversely, if $\mathscr{U}$ is a $\mathcal{C}_{\mathscr{V}}$-module then it is a $\mathscr{V}$-module since $\mathsf{H}_{n,d}$ acts trivially on the image of $\psi_{N-1}\cdots\psi_d$ by definition of a functor. Furthermore, if $\mathscr{W}, \mathscr{U}$ are $\mathscr{C}_{\mathscr{V}}$-modules, then a morphism of functors $\mathscr{W} \to \mathscr{U}$ (also called a natural transformation) coincides with a morphism of sequences in Definition 2.1. Applying the constructions in Section 2.3 to $\mathcal{C}$-modules yields other $\mathcal{C}$-modules.

Example A.3. *Here are some examples of the categories resulting from Definition A.2.*

(a) The category corresponding to Example 1.5 with $\mathsf{G}_n = \mathsf{S}_n$ is (the skeleton of) $\mathcal{C} = \mathrm{FI}$, the category whose objects are finite sets and whose morphisms are injections.

(b) The category corresponding to Example 1.5 with $\mathsf{G}_n = \mathsf{B}_n$ (resp., D_n) is $\mathcal{C} = \mathrm{FI}_{BC}$ (resp., $\mathcal{C} = \mathrm{FI}|_D$) defined in [50], whose objects are the sets $[\pm n] \coloneqq \{\pm 1, \ldots, \pm n\}$ for $n \in \mathbb{N}$ and whose morphisms are injections $f\colon [\pm n] \hookrightarrow [\pm N]$ satisfying $f(-i) = -f(i)$ (and reverse evenly-many signs if $\mathsf{G}_n = \mathsf{D}_n$).

(c) The category corresponding to the graphon sequence in Section 4.7 is the opposite category $\mathcal{C} = \mathcal{P}_2^{\mathrm{op}}$ of the category $\mathcal{P}_2$ with objects $[2^n]$ and morphisms which are 2^{N-n}-to-one surjections $[2^N] \to [2^n]$, or equivalently, partitions of $[2^N]$ into 2^n equal parts.

Following [50], we say $\mathcal{C} = \mathrm{FI}|_{\mathcal{W}}$ if $\mathcal{C} = \mathrm{FI}, \mathrm{FI}|_{BC}$ or $\mathrm{FI}|_D$. (Algebraically) free $\mathcal{C}$-modules are defined exactly as in Definition 2.6, see [49, Def. 2.2.2] and [51, Def. 1.8,3.1] for example. The theory of [51] gives the following result for $\mathcal{C} = \mathrm{FI}|_{\mathcal{W}}$, which extends to categories of FI-type introduced in [51].

Theorem A.4 ([51, Thm. B(1)])**.** *Tensor products of free $\mathrm{FI}|_{\mathcal{W}}$-modules are free.*

The following result illustrates two further properties of $\mathrm{FI}|_{\mathcal{W}}$-modules, the second of which is included in our Theorem 2.11 stated above.

Theorem A.5 (Noetherianity and tensor products)**.** *Let $\mathcal{C} = \mathrm{FI}|_{\mathcal{W}}$.*

(Noetherianity) *Any submodule of a finitely-generated $\mathcal{C}$-module is finitely-generated.*

(Tensor products) *If $\mathscr{V}_1$ and $\mathscr{V}_2$ are $\mathcal{C}$-modules generated in degrees d_1 and d_2, respectively, then $\mathscr{V}_1 \otimes \mathscr{V}_2$ is generated in degree $d_1 + d_2$.*

Proof. Noetherianity is shown for FI in [49, Thm. 1.13] and for $\mathrm{FI}|_{BC}, \mathrm{FI}|_D$ in [50, Thm. 4.21]. The generation degree bound is shown in [49, Prop. 2.3.6] for FI and in [50, Prop. 5.2] for $\mathrm{FI}|_{BC}, \mathrm{FI}|_D$. □

Noetherianity helps explain the ubiquity of representation stability, while the generation degree bound for tensor products allows us to bound the generation degrees of complicated sequences from degrees of simple ones. The two properties in Theorem A.5 hold over more general categories than $\mathrm{FI}|_{\mathcal{W}}$, including for categories of FI-type and certain quasi-Gröbner categories introduced in [54]. We remark that these two types of categories only include representations of finite groups, and do not include the graphon category in Example A.3(c), whose properties would be interesting to study in future work.

Definition A.6 (Property (TFG))**.** *We say that a category $\mathcal{C}$ satisfies* property (TFG) *if Tensor products of free $\mathcal{C}$-modules are Free and satisfy the Generation degree bound in Theorem A.5.*

An example of a category not satisfying (TFG) is given in [54, Rmk. 7.4.3]. We can use property (TFG) to obtain a calculus for presentation degrees from which Theorem 2.11 may be deduced.

Proposition A.7. *Suppose $\mathcal{C}$ is a category satisfying (TFG). If $\mathscr{V},\mathscr{U}$ are $\mathcal{C}$-modules which are generated in degrees d_V, d_U and presented in degrees k_V, k_U, respectively, then $\mathscr{V}\otimes\mathscr{U}$ is presented in degree $\max\{d_V + k_U, d_U + k_V\}$.*

Proof. Suppose $\mathscr{F}_V,\mathscr{F}_U$ are free $\mathcal{C}$-modules generated in degrees d_V, d_U, respectively, and $\mathscr{F}_V\to\mathscr{V}$ and $\mathscr{F}_U\to\mathscr{U}$ are surjective morphisms whose kernels $\mathscr{K}_V$ and $\mathscr{K}_U$ are generated in degrees r_V, r_U, respectively. Then $\mathscr{F}_V\otimes\mathscr{F}_U$ is a free $\mathcal{C}$-module generated in degree d_V+d_U by (TFG), and the morphism $\mathscr{F}_V\otimes\mathscr{F}_U\to\mathscr{V}\otimes\mathscr{U}$ is surjective with kernel $\mathscr{K}_V\otimes\mathscr{F}_U+\mathscr{F}_V\otimes\mathscr{K}_U$. Since $\mathscr{K}_V\otimes\mathscr{F}_U$ is generated in degree r_V+d_U and similarly for $\mathscr{F}_U\otimes\mathscr{K}_V$, their sum is generated in degree $\max\{r_V+d_U, d_V+r_U\}$. □

Our next goal is to understand presentation degrees for images, and in particular, for Schur functors. We begin with a number of elementary lemmas.

Lemma A.8. *Let $\mathscr{V}$ and $\mathscr{U}$ be $\mathcal{C}$-modules. If $\mathscr{V},\mathscr{U}$ are generated in degrees d_V, d_U and presented in degrees k_V, k_U, respectively, then $\mathscr{V}\oplus\mathscr{U}$ is generated in degree $\max\{d_V, d_U\}$ and presented in degree $\max\{k_V, k_U\}$.*

Proof. The claim about the generation degree is immediate from its definition. Suppose that $\mathscr{F}_V\to\mathscr{V}$ and $\mathscr{F}_U\to\mathscr{U}$ are surjective morphisms with $\mathscr{F}_V,\mathscr{F}_U$ being free $\mathcal{C}$-modules generated in degrees d_V, d_U with kernels $\mathscr{K}_V,\mathscr{K}_U$ generated in degrees k_V, k_U, respectively. Then $\mathscr{F}_V\oplus\mathscr{F}_U$ is free (by definition) and generated in degree $\max\{d_V, d_U\}$, and surjects onto $\mathscr{V}\oplus\mathscr{U}$ with kernel $\mathscr{K}_V\oplus\mathscr{K}_U$ which is generated in degree $\max\{k_V, k_U\}$. □

Lemma A.9. *Let $\mathscr{V}=\{\mathbb{V}_n\}$ and $\mathscr{U}=\{\mathbb{U}_n\}$ be two $\mathcal{C}$-modules, let $\mathscr{A}=\{A_n\}\colon\mathscr{V}\to\mathscr{U}$ be a surjective morphism, and let $\mathscr{W}=\{\mathbb{W}_n\subseteq\mathbb{U}_n\}$ be a $\mathcal{C}$-submodule of $\mathscr{U}$. If $\ker\mathscr{A}$ is generated in degree d and $\mathscr{W}$ is generated in degree d_W, then $\mathscr{A}^{-1}(\mathscr{W})=\{A_n^{-1}(\mathbb{W}_n)\}$ is a $\mathcal{C}$-module generated in degree $\max\{d, d_W\}$.*

Proof. Define the consistent sequence $\mathbb{Z}_n=\mathbb{R}[\mathsf{G}_n]\left(A_{d_W}^{\dagger}\mathbb{W}_{d_W}\right)\subseteq\mathbb{V}_n$ if $n\geq d_W$ and $\mathbb{Z}_n=0$ otherwise, where $A_{d_W}^{\dagger}$ is the pseudoinverse of A_{d_W}. Note that $\{\mathbb{Z}_n\}$ is generated in degree d_W. Moreover, $A_n^{-1}(\mathbb{W}_n)=\ker A_n+\mathbb{Z}_n$. Indeed, we have $A_nA_{d_W}^{\dagger}=A_{d_W}A_{d_W}^{\dagger}=\mathrm{id}_{\mathbb{U}_{d_W}}$ because $\{A_n\}$ is a surjective morphism, hence $A_n(\ker A_n+\mathbb{Z}_n)=A_n(\mathbb{Z}_n)=\mathbb{R}[\mathsf{G}_n]\mathbb{W}_{d_W}=\mathbb{W}_n$. Conversely, if $A_nx\in\mathbb{W}_n=\mathbb{R}[\mathsf{G}_n]\mathbb{W}_{d_W}$ then we can write $A_nx=\sum_i g_iw_i$ for $g_i\in\mathsf{G}_n$ and $w_i\in\mathbb{W}_{d_W}$. Then $\hat{x}=\sum_i g_iA_{d_W}^{\dagger}w_i\in\mathbb{Z}_n$ and $A_n(x-\hat{x})=0$, so $x\in\ker A_n+\mathbb{Z}_n$. Since $\ker\mathscr{A}$ is generated in degree d and $\{\mathbb{Z}_n\}$ is generated in degree d_W, their sum is generated in degree $\max\{d, d_W\}$. □

Lemma A.10. *Suppose $\mathscr{V}=\{\mathbb{V}_n\}$ and $\{\mathbb{U}_n\}$ are two $\mathcal{C}$-modules, and $\mathscr{A}=\{A_n\}\colon\mathscr{V}\to\mathscr{U}$ is a morphism. If $\mathscr{V}$ is generated in degree d, then $\mathrm{im}\mathscr{A}$ is generated in degree d. If, moreover, $\mathscr{A}^*=\{A_n^*\}$ is a morphism, then $\ker\mathscr{A}$ is also generated in degree d.*

Proof. The first claim follows from $A_n(\mathbb{V}_n)=A_n(\mathbb{R}[\mathsf{G}_n]\mathbb{V}_d)=\mathbb{R}[\mathsf{G}_n]A_n(\mathbb{V}_d)=\mathbb{R}[\mathsf{G}_n]A_d(\mathbb{V}_d)$, where we used the equivariance of A_n and the fact that $A_n|_{\mathbb{V}_d}=A_d$. For the second claim, note that if $\mathscr{A}^*$ is a morphism, then $\{\mathrm{im}\,A_n^*=(\ker A_n)^{\perp}\}$ is a $\mathcal{C}$-submodule of $\mathscr{V}$. Therefore, $\{\mathcal{P}_{\ker A_n}\}\colon\mathscr{V}\to\mathscr{V}$ is a morphism, and its image is precisely $\ker\mathscr{A}$. □

Proposition A.11. *Suppose $\mathscr{V}=\{\mathbb{V}_n\},\mathscr{U}=\{\mathbb{U}_n\}$ are two $\mathcal{C}$-modules and both $\mathscr{A}=\{A_n\colon\mathbb{V}_n\to\mathbb{U}_n\}$ and $\{A_n^*\colon\mathbb{U}_n\to\mathbb{V}_n\}$ are morphisms. If $\mathscr{V}$ is generated in degree d and presented in degree k, then $\mathrm{im}\mathscr{A}=\{A_n(\mathbb{V}_n)\}$ is generated in degree d and presented in degree k.*

Proof. Let $\mathscr{F}=\{\mathbb{F}_n\}$ be a free $\mathcal{C}$-module generated in degree d and let $\mathscr{B}=\{B_n\}\colon\mathscr{F}\to\mathscr{V}$ be a surjective morphism whose kernel $\mathscr{K}=\{\mathbb{K}_n\}$ is generated in degree k. The composition $\mathscr{F}\xrightarrow{\mathscr{B}}\mathscr{V}\xrightarrow{\mathscr{A}}\mathrm{im}\mathscr{A}$ is a surjective morphism from the free $\mathcal{C}$-module $\mathscr{F}$ whose kernel is $\mathscr{B}^{-1}(\ker\mathscr{A})$ and is generated in degree $\max\{d,k\}=k$ by Lemmas A.9 and A.10. □

Corollary A.12. *Suppose $\mathscr{C}$ satisfies property (TFG). If $\mathscr{V}$ is a $\mathcal{C}$-module generated in degree d and presented in degree k, and λ is a partition, then $\mathbb{S}^{\lambda}\mathscr{V}$ is generated in degree $d|\lambda|$ and presented in degree $k+d(|\lambda|-1)$.*

Schur functors generalize symmetric and alternating algebras, see [62, §6.1]. Their generation degree for $\mathcal{C} = \mathrm{FI}$ was bounded using a similar approach in [49, Prop. 3.4.3].

Proof. By Proposition A.7, the $\mathcal{C}$-module $\mathscr{V}^{\otimes|\lambda|}$ is generated in degree $d|\lambda|$ and presented in degree $d(|\lambda|-1)+k$. Let $\mathsf{S}_{|\lambda|}$ act on each $\mathbb{V}_n^{\otimes|\lambda|}$ by permuting its factors $\sigma\cdot(v_1\otimes\cdots\otimes v_{|\lambda|}) = v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(|\lambda|)}$, which is an orthogonal action commuting with the emebddings $\mathbb{V}_n^{\otimes|\lambda|}\subseteq\mathbb{V}_{n+1}^{\otimes|\lambda|}$. In this way, any element $c_\lambda\in\mathbb{R}[\mathsf{S}_{|\lambda|}]$ defines a morphism $c_\lambda\colon\mathscr{V}^{\otimes|\lambda|}\to\mathscr{V}^{\otimes|\lambda|}$ such that $c_\lambda^*\in\mathbb{R}[\mathsf{S}_{|\lambda|}]$ is also a morphism. If $c_\lambda\in\mathbb{R}[\mathsf{S}_{|\lambda|}]$ is the Young symmetrizer corresponding to partition λ, then $\operatorname{im} c_\lambda = \mathbb{S}^\lambda\mathscr{V}$. The result follows from Proposition A.11. □

We conclude this appendix by summarizing our calculus for generation and presentation degrees. Instantiating the following theorem with $\mathcal{C} = \mathrm{FI}|_{\mathcal{W}}$ yields Theorem 2.11.

Theorem A.13 (Calculus for generation and presentation degrees)**.** *Let $\mathscr{V}=\{\mathbb{V}_n\},\mathscr{U}=\{\mathbb{U}_n\}$ be $\mathcal{C}$-modules generated in degrees d_V, d_U and presented in degrees k_V, k_U, respectively.*

- (Sums) *$\mathscr{V}\oplus\mathscr{U}$ is generated in degree $\max\{d_V,d_U\}$ and presented in degree $\max\{k_V,k_U\}$.*
- (Images and kernels) *If $\mathscr{A}\colon\mathscr{V}\to\mathscr{U}$ and $\mathscr{A}^*$ are morphisms, then $\operatorname{im}\mathscr{A}$ and $\ker\mathscr{A}$ are generated in degree d_V and presented in degree k_V.*

Suppose $\mathcal{C}$ satisfies (TFG). Then

- (Tensors) *$\mathscr{V}\otimes\mathscr{U}$ is generated in degree d_V+d_U and presented in degree $\max\{k_V+d_U, k_U+d_V\}$.*
- (Schur functors) *$\mathbb{S}^\lambda\mathscr{V}$ is generated in degree $d_V|\lambda|$ and presented in degree $d_V(|\lambda|-1)+k_V$ for any partition λ.*

B Deriving the Presentation Degree from Extendability

In this appendix, we show how the definitions of algebraically free consistent sequences and the presentation degree naturally arise when trying to extend a fixed equivariant linear map to a morphism of sequences, necessary for our algorithm in Section 6.

Let $\mathscr{V}=\{\mathbb{V}_n\}$, $\mathscr{U}=\{\mathbb{U}_n\}$ be consistent sequences of $\{\mathsf{G}_n\}$-representations, fix $n_0\in\mathbb{N}$ and consider a linear map $A_{n_0}\in\mathcal{L}(\mathbb{V}_{n_0},\mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$. When can we extend A_{n_0} to a morphism of sequences $\{A_n\}$? We seek conditions on A_{n_0} which are easy to enforce computationally, so that they can be used to parametrize and search over compatible sequences of convex sets computationally. The following proposition gives an equivalent characterization for the existence of such an extension.

Proposition B.1. *Let $\mathscr{V}=\{\mathbb{V}_n\},\mathscr{U}=\{\mathbb{U}_n\}$ be consistent sequences such that $\mathscr{V}$ is generated in degree d, and fix $A_{n_0}\in\mathcal{L}(\mathbb{V}_{n_0},\mathbb{U}_{n_0})^{\mathsf{G}_{n_0}}$ for $n_0\geq d$.*

(a) There exists $\{A_n\in\mathcal{L}(\mathbb{V}_n,\mathbb{U}_n)^{\mathsf{G}_n}\}_{n<n_0}$ satisfying $A_{n+1}|_{\mathbb{V}_n}=A_n$ for all $n<n_0$ if and only if $A_{n_0}(V_j)\subseteq U_j$ for $j\leq d$.

(b) There exists $\{A_n\in\mathcal{L}(\mathbb{V}_n,\mathbb{U}_n)^{\mathsf{G}_n}\}_{n>n_0}$ satisfying $A_{n+1}|_{\mathbb{V}_n}=A_n$ for all $n\geq n_0$ if and only if the following implication holds

$$\sum\nolimits_i g_ix_i=0 \implies \sum\nolimits_i g_iA_{n_0}x_i=0, \qquad \text{for all } g_i\in\mathsf{G}_n,\ x_i\in\mathbb{V}_d,\ n\in\mathbb{N}. \tag{32}$$

If an extension $\{A_n\}$ of A_{n_0} exists, then it is unique.

Proof. (a) If such $\{A_n\}_{n<n_0}$ exists, then it is uniquely given in terms of A_{n_0} by $A_n = A_{n_0}|_{\mathbb{V}_n}$. Therefore, we have $A_{n_0}(V_j)=A_j(V_j)\subseteq U_j$ for all $j\leq d$. Conversely, suppose $A_{n_0}(V_j)\subseteq U_j$ for $j\leq d$. We claim that $A_{n_0}(\mathbb{V}_n)\subseteq\mathbb{U}_n$ for all $d\leq n\leq n_0$ as well. Indeed, because $\mathscr{V}$ is generated in degree d, we have $A_{n_0}(\mathbb{V}_n)=A_{n_0}(\mathbb{R}[\mathsf{G}_n]\mathbb{V}_d)=\mathbb{R}[\mathsf{G}_n]A_{n_0}(\mathbb{V}_d)\subseteq\mathbb{R}[\mathsf{G}_n]\mathbb{U}_d\subseteq\mathbb{U}_n$ for such $n\geq d$. Defining $A_n=A_{n_0}|_{\mathbb{V}_n}$ for each $n<n_0$ yields the desired extension to lower dimensions.

(b) If such $\{A_n\}_{n>n_0}$ exists, it is unique and is explicitly given in terms of A_{n_0} as follows. For any $n > n_0$ and $x \in \mathbb{V}_n$, we can write $x = \sum_i g_i x_i$ for some $g_i \in \mathsf{G}_n$ and $x_i \in \mathbb{V}_d \subseteq \mathbb{V}_{n_0}$ by definition of the generation degree. Because $A_n \colon \mathbb{V}_n \to \mathbb{U}_n$ is G_n-equivariant and satisfies $A_n|_{\mathbb{V}_{n_0}} = A_{n_0}$, we have

$$A_n x = A_n \left(\sum\nolimits_i g_i x_i\right) = \sum\nolimits_i g_i(A_{n_0} x_i), \tag{33}$$

which expresses A_n in terms of A_{n_0}. The expression (33) shows that (32) is satisfied. Conversely, suppose that (32) is satisfied. For each $n > n_0$ define $A_n \colon \mathbb{V}_n \to \mathbb{U}_n$ as follows. For any $x \in \mathbb{V}_n$, write $x = \sum_i g_i x_i$ for some $g_i \in \mathsf{G}_n$ and $x_i \in \mathbb{V}_d$, which is possible because $\mathscr{V}$ is generated in degree d, and set $A_n x$ to the right-hand side of (33). This is well-defined because if $x = \sum_i g_i x_i = \sum_j g'_j x'_j$ for $g_i, g'_j \in \mathsf{G}_n$ and $x_i, x'_j \in \mathbb{V}_d$ then $\sum_i g_i A_{n_0} x_i = \sum_j g'_j A_{n_0} x'_j$ by (32). Moreover, A_n is linear, G_n-equivariant, and extends A_{n_0} by construction, so $\{A_n\}_{n>n_0}$ is the desired extension of A_{n_0}. □

The conditions on A_{n_0} in Proposition B.1(a) are easy to impose computationally, and we do so in Section 6.1. In contrast, while condition (32) in Proposition B.1(b) fully characterizes extendability of A_{n_0} to higher dimensions, it is unclear how to impose it computationally. We therefore proceed to study it further. In algebraic terms, elements $x_i \in \mathbb{V}_d$ are called the *generators* of $\mathscr{V}$, and expressions of the form $\sum_i g_i x_i = 0$ with $g_i \in \mathsf{G}_n$ are called *relations* between those generators over the group G_n. Proposition B.1(b) shows that A_{n_0} extends to a morphism iff any relation satisfied by the generators of $\mathscr{V}$ is also satisfied by their images under A_{n_0}. We therefore need to understand the relations among the generators in $\mathbb{V}_d$.

We study these relations in two stages. First, we identify two simple types of relations that are satisfied by the images of the generators under A_{n_0} for appropriate $\mathscr{V}, \mathscr{U}$. We then define algebraically free consistent sequences whose generators satisfy only these two types of relations. Second, we express an arbitrary consistent sequence as the quotient of an algebraically free one. The kernel of this quotient morphism is a consistent sequence that encodes all additional relations. To capture the degree starting from which both the generators and the relations between them stabilize, we define the presentation degree of a consistent sequence as the maximum of the generation degree of the sequence itself and that of the above kernel, see Definition 2.7. The presentation degree plays a prominent role in our structural results in Section 3.

We begin carrying out the first stage of the above program and identify two simple types of relations. The first source for relations between generators in $\mathbb{V}_d$ arises from relations over G_d. Indeed, if $\sum_i g_i x_i = 0$ for $g_i \in \mathsf{G}_d$ and $x_i \in \mathbb{V}_d$ then $\sum_i g g_i x_i = 0$ for any $g \in \mathsf{G}_n$. Such relations are always satisfied by the images $A_{n_0} x_i$. A second source for such relations arises from subgroups of G_n acting trivially on $\mathbb{V}_d$, which are presicely the centralizing subgroups of Definition 2.4. Centralizing subgroups yield a second source for relations, namely, the relations $(h - \mathrm{id})x = 0$ for all $x \in \mathbb{V}_d$ and $h \in \mathsf{H}_{n,d}$. Thus, if A_{n_0} extends to a morphism then $A_{n_0}(\mathbb{V}_d) \subseteq \bigcap_{n\geq d} \mathbb{U}_n^{\mathsf{H}_{n,d}}$. Rather than attempt to enforce these constraints computationally, we make a standard simplifying assumption from the representation stability literature. Specifically, we assume that $\mathbb{U}_d$ is fixed by $\mathsf{H}_{n,d}$ (or a subgroup of it, see below) for all $n \geq d$, in which case there is a simple sufficient condition for the above constraints that we can impose computationally. This is precisely the assumption that $\mathscr{U}$ is a $\mathscr{V}$-module as in Definition 2.5. This terminology comes from a categorical approach to representation stability, see Appendix A. If $\mathscr{U}$ is a $\mathscr{V}$-module, then imposing $A_{n_0}(\mathbb{V}_d) \subseteq \mathbb{U}_d$ is sufficient to guarantee $A_{n_0}(\mathbb{V}_d) \subseteq \bigcap_{n\geq d} \mathbb{U}_n^{\mathsf{H}_{n,d}}$ since $\mathbb{U}_d$ is contained in the right-hand side of this inclusion. Imposing this sufficient condition can done computationally, as we do in Section 6.1. This concludes the first stage.

To go beyond the above two simple types of relations satisfied by the generators, we define algebraically free consistent sequences (Definition 2.6), whose generators do not satisfy any additional types of relations. We then write any consistent sequence as the image under a morphism of sequences of an algebraically free one. The kernel of this morphism precisely captures the relations satisfied by the generators. To that end, note that any finitely-generated consistent sequence is the image under a morphism of sequences of an algebraically free sequence.

Proposition B.2. *Let $\mathscr{V}$ be a consistent sequence of $\{\mathsf{G}_n\}$-representations and let $\mathscr{U} = \{\mathbb{U}_n\}$ be a $\mathscr{V}$-module. Then $\mathscr{U}$ is generated in degree d if and only if there exists an algebraically free $\mathscr{V}$-module $\mathscr{F}$ generated in degree d and a surjective morphism of sequences $\mathscr{F} \to \mathscr{U}$.*

Proof. If $\mathscr{F} = \{\mathbb{F}_n\}$ is a $\mathscr{V}$-module and $\{A_n \colon \mathbb{F}_n \to \mathbb{U}_n\}$ is a surjective morphism, then for any $n \geq d$ we have $\mathbb{U}_n = A_n(\mathbb{F}_n) = A_n(\mathbb{R}[\mathsf{G}_n]\mathbb{F}_d) = \mathbb{R}[\mathsf{G}_n]A_n(\mathbb{F}_d) = \mathbb{R}[\mathsf{G}_n]A_d(\mathbb{F}_d) = \mathbb{R}[\mathsf{G}_n]\mathbb{U}_d$, where we used the fact that

$\mathscr{F}$ is generated in degree d; the equivariance of A_n; the fact that $A_n|_{\mathbb{F}_d} = A_d$ since $\{A_n\}$ is a morphism; and the surjectivity of A_d. This shows $\mathscr{U}$ is generated in degree d.

Conversely, if $\mathscr{U}$ is generated in degree d, define the algebraically free $\mathscr{V}$-module $\mathscr{F} = \bigoplus_{i=1}^d \mathrm{Ind}_{\mathsf{G}_i}(U_i)$ and consider the morphism $\mathscr{F} \to \mathscr{U}$ defined by $g \otimes u \mapsto g \cdot u$ for each $g \in \mathsf{G}_n$, $u \in U_i$, and $i \in [d]$ (see Section 1.3). The image of this morphism in $\mathbb{U}_n$ is precisely $\sum_{i=1}^{\min\{d,n\}} \mathbb{R}[\mathsf{G}_n]U_i$, hence it is surjective for all n. Finally, $\mathrm{Ind}_{\mathsf{G}_i}(U_i)$ is generated in degree i, so $\mathscr{F}$ is generated in degree d. □

The kernel of the morphism in Proposition B.2 precisely encodes all the additional relations beyond the two simple types above satisfied by the generators of $\mathscr{V}$. The generation degree of this kernel then captures the point at which relations stabilize. We therefore define the presentation degree (Definition 2.7) as the maximum of the generation degree of $\mathscr{V}$ and that of this kernel, which captures stabilization of the generators as well as of the relations between them.

The presentation degree allows us to ensure condition (32) is satisfied and hence to extend a fixed linear map to a morphism of sequences in Theorem 3.4, thus answering the question posed in the beginning of this section. Indeed, comparing Theorem 3.4 with Proposition B.1, we see that condition (32) is satisfied by any fixed equivariant map in dimension n_0 exceeding the presentation degree. Thus, the presentation degree appears in our extendability result for convex sets (Theorem 3.5), and in our algorithm for computationally parametrizing such sets (Algorithm 1).

C Schur-Horn Orbitopes in AF Algebras

The permutahedra and Schur-Horn orbitopes studied in Section 4.3 are special cases of a more general construction arising in the field of operator algebras, and which is naturally analyzed using our framework. In this appendix, we present this unified view. Let $\mathbb{A}_n \subseteq \mathbb{M}_n := \mathbb{C}^{m2^n \times m2^n}$ be a unital $\mathbb{C}$-subalgebra with the faithful tracial state $\tau_n(x) = \mathrm{Tr}(x)/m2^n$ which induces the inner product $\langle x, y\rangle_n = \tau_n(x^*y)$, and the max-singular value operator norm $\|\cdot\|_{(n)}$. Let $\varphi_n \colon \mathbb{M}_n \hookrightarrow \mathbb{M}_{n+1}$ be the unital algebra embedding $\varphi_n(x) = x \otimes I_2$, and note that $\|x \otimes I_2\|_{(n+1)} = \|x\|_{(n)}$ and $\tau_{n+1} \circ \varphi_n = \tau_n$, so φ_n is an isometry with respect to the above inner product. Thus, both the operator norm $\|\cdot\|$ and the normalized trace τ do not depend on n and extend to the limit $\mathbb{A}_\infty = \bigcup_n \mathbb{A}_n$, and we omit their subscripts above. Let $\overline{\mathbb{A}_\infty}$ be the closure of $\mathbb{A}_\infty$ with respect to the operator norm, which is now a C^* algebra called an approximately finite-dimensional (AF) algebra [89, Chap. 3]. Note that τ extends continuously to $\overline{\mathbb{A}_\infty}$ since $|\tau(x)\| \leq \|x\|$ for all $x \in \mathbb{A}_\infty$, and hence τ defines a tracial state on $\overline{\mathbb{A}_\infty}$.

The spectral theorem for C^* algebras gives an (isometric) isomorphism of algebras between continuous functions on the spectrum $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda I \text{ not invertible}\}$ of any self-adjoint $x \in \overline{\mathbb{A}_\infty}$ and the closed subalgebra it generates [90, Thm. 11.19]. This allows us to apply continuous functions $f \colon \sigma(x) \to \mathbb{C}$ to x itself to obtain $f(x) \in \overline{\mathbb{A}_\infty}$. Using this functional calculus, for each self-adjoint $x \in \overline{\mathbb{A}_\infty}$ we get the positive unital linear functional on such functions sending $f \mapsto \tau(f(x))$. By the Riesz representation theorem [91, Thm. 6.19], there is a unique probability measure μ_x^τ on $\sigma(x)$ satisfying $\mu_x^\tau(f) = \tau(f(x))$ for all continuous $f \colon \sigma(x) \to \mathbb{C}$, which is called the spectral measure of x with respect to τ. Finally, if $\sigma(x) = \{\lambda_1, \ldots, \lambda_q\}$ is finite then we have a spectral decomposition $x = \sum_{i=1}^q \lambda_i p_i$ where $\{p_i\}$ are orthogonal projectors satisfying $p_i p_j = \delta_{i,j} p_i$, $\sum_i p_i = 1$, and $\mu_x^\tau = \sum_{i=1}^q \tau(p_i)\delta_{\lambda_i}$.

Let $\mathbb{V}_n = \mathbb{A}_n \cap \mathbb{M}_n^{\mathrm{s.a.}}$ be the (finite-dimensional real) vector space of self-adjoint elements in $\mathbb{A}_n$, and note that $\varphi_n(\mathbb{V}_n) \subseteq \mathbb{V}_{n+1}$ and that $\overline{\mathbb{V}_\infty}$ is the collection of self-adjoints in $\overline{\mathbb{A}_\infty}$. Let $\mathsf{G}_n \subseteq \mathrm{U}(m2^n)$ be a group of unitaries acting on $\mathbb{V}_n$ by conjugation such that $\mathsf{G}_n \subseteq \mathsf{G}_{n+1}$ under the embedding $g \mapsto g \otimes I_2$, so that $\{\mathbb{V}_n\}$ is a consistent sequence of $\{\mathsf{G}_n\}$-representations.

Lemma C.1. *Let $\mathcal{P}_n \colon \mathbb{V}_\infty \to \mathbb{V}_n$ be orthogonal projections. Then $\|\mathcal{P}_n x\| \leq \|x\|$ and $\tau \circ \mathcal{P}_n = \tau$ for all n.*

Proof. Note that $\varphi_n^* \colon \mathbb{V}_{n+1} \to \mathbb{V}_n$ satisfies

$$\varphi_n^*(x) = \frac{1}{2}\sum_{i=1}^{2} (I \otimes e_i)^\top x (I \otimes e_i), \qquad \text{hence } \|\varphi_n^* x\| \leq \|x\|, \tag{34}$$

for all $x \in \mathbb{V}_{n+1}$. Because the orthogonal projection $\mathcal{P}_n\colon \mathbb{V}_\infty \to \mathbb{V}_n$ satisfies $\mathcal{P}_n|_{\mathbb{V}_N} = \varphi_n^* \circ \ldots \circ \varphi_{N-1}^*$ for each $N > n$, we conclude that $\|\mathcal{P}_n x\| \leq \|x\|$ for all $x \in \mathbb{V}_\infty$. Also, for any $x \in \mathbb{V}_\infty$ we have $\tau(\mathcal{P}_n x) = \langle \mathcal{P}_n x, 1\rangle = \langle x, \mathcal{P}_n 1\rangle = \langle x, 1\rangle = \tau(x)$, hence the second claim follows. ☐

In particular, the norm $\|\cdot\|$ satisfies the condition of Definition 2.21. Let $\lambda \in \mathbb{R}^q$ and $\widetilde{\lambda} = [\lambda_1 \mathbb{1}_{m_i}^\top, \ldots, \lambda_q \mathbb{1}_{m_q}^\top]^\top \in \mathbb{R}^m$ has entry λ_i repeated m_i times (so $m = \sum_i m_i$). Then $x_1 = \mathrm{diag}(\widetilde{\lambda})$ has spectral measure $\mu_{x_1}^\tau = \sum_{i=1}^q \frac{m_i}{m}\delta_{\lambda_i}$. The Shur-Horn orbitopes associated to x_1 is then

$$\mathrm{SH}(x_1)_n = \mathrm{conv}\{x \in \mathbb{V}_n : \mu_x^\tau = \mu_{x_1}^\tau\} = \mathrm{conv}\left\{\sum\nolimits_{i=1}^q \lambda_i p_i : p_i \in \mathbb{V}_n,\ p_i p_j = \delta_{i,j} p_i,\ \sum\nolimits_i p_i = 1,\ \tau(p_i) = \frac{m_i}{m}\right\},$$

To obtain conic descriptions for the above orbitopes, let $\mathcal{K}_n = \{xx^* : x \in \mathbb{A}_n\} \subseteq \mathbb{V}_n$ be the cone of positive-semidefinite Hermitian matrices in $\mathbb{A}_n$, so that $\mathcal{K}_n \subseteq K_{n+1}$ and $\overline{\mathcal{K}_\infty} = \{xx^* : x \in \overline{\mathbb{A}_\infty}\}$. We also have $\mathcal{P}_n(K_{n+1}) = \mathcal{K}_n$ by (34), hence $\mathcal{P}_n(\overline{\mathcal{K}_\infty}) = \mathcal{K}_n$. Therefore, the finite-dimensional description of Schur-Horn orbitopes [70, Eq. (19)] reads

$$\mathrm{SH}(x_1)_n = \left\{\sum\nolimits_{i=1}^q \lambda_i p_i : p_i \in \mathcal{K}_n,\ \sum\nolimits_i p_i = 1,\ \tau(p_i) = m_i/m\right\}. \tag{35}$$

Our Theorem 3.6 now yields the following infinite-dimensional extension of these descriptions.

Proposition C.2. *Let $\lambda_1, \ldots, \lambda_q$ are distinct real numbers, and let $m_1, \ldots, m_q \in \mathbb{N}$ with sum $m = \sum_i m_i$. Then $\overline{\mathrm{SH}(x_1)_\infty}$ with $x_1 = \mathrm{diag}(\widetilde{\lambda})$ as above is*

$$\overline{\mathrm{conv}}\left\{x \in \overline{\mathbb{V}_\infty} : \mu_x^\tau = \sum\nolimits_{i=1}^q \frac{m_i}{m}\delta_{\lambda_i}\right\} = \overline{\left\{\sum\nolimits_{i=1}^q \lambda_i p_i : p_i \in \overline{\mathcal{K}_\infty},\ \sum\nolimits_i p_i = 1,\ \tau(p_i) = m_i/m\right\}}. \tag{36}$$

Proof. First, the left-hand side of (36) is just $\overline{\mathrm{SH}(x_1)_\infty}$. Indeed, if $x_n \in \overline{\mathbb{V}_\infty}$ is a sequence converging to x with $\mu_{x_n}^\tau = \mu_{x_1}^\tau$ for all n, then for any continuous $f\colon \mathbb{R} \to \mathbb{C}$ we have $\mu_x^\tau(f) = \tau(f(x)) = \lim_n \tau(f(x_n)) = \mu_{x_1}^\tau(f)$ so $\mu_x^\tau = \mu_{x_1}^\tau$. Conversely, if $\mu_x^\tau = \sum_i \frac{m_i}{m}\delta_{\lambda_i}$ then x admits a decomposition $x = \sum_i \lambda_i p_i$ with $\tau(p_i) = m_i/m$. Define $x_n = \mathcal{P}_n x = \sum_{i=1}^q \lambda_i(\mathcal{P}_n p_i)$ and note that $\mathcal{P}_n p_i \in \mathcal{K}_n$ for all i, that $\sum_i \mathcal{P}_n p_i = \mathcal{P}_n 1 = 1$, and that $\tau(\mathcal{P}_n p_i) = \tau(p_i) = m_i/m$. Thus, $x_n \in \mathrm{SH}(x_1)_n$ for all n and $x_n \to x$ by Lemma 2.22, hence $x \in \overline{\mathrm{SH}(x_1)_\infty}$.

Second, we appeal to Theorem 3.6 to obtain equality in (36). Note that the conic description (35) is of the form (ConicSeq) with $\mathscr{V} = \{\mathbb{V}_n\}, \mathscr{W} = \mathscr{V}^{\oplus q}, \mathscr{U} = \mathscr{W} \oplus \mathscr{V}^{\oplus 2} \oplus \mathbb{R}^q$, and

$$A_n x = (0, -x, 0, 0), \quad B_n(p_1, \ldots, p_q) = \left(p_1, \ldots, p_q, \sum\nolimits_i \lambda_i p_i, \sum\nolimits_i p_i, \tau(p_1), \ldots, \tau(p_q)\right),$$

and $u_n = (0, 0, -I, -m_1/m, \ldots, -m_q/m)$. Observing that $\{A_n\}, \{A_n^*\}, \{B_n\}, \{B_n^*\}$ are all morphisms, that $u_n = u_{n+1}$ (under our embeddings), the descriptions in (35) are free and satisfy the hypohteses of Proposition 3.2(a). Furthermore, putting the norm $\|(p_1, \ldots, p_q)\| = \max_i \|p_i\|$ on W_∞ and $\|(p_1, \ldots, p_q, y_1, y_2, v)\| \leq \max\{\|p_i\|, \|y_j\|, \|v\|_\infty\}$ on $\mathbb{U}_\infty$, we have $\overline{W_\infty} = \overline{\mathbb{V}_\infty}^{\oplus q}$ and $\overline{\mathbb{U}_\infty} = \overline{W_\infty} \oplus \overline{\mathbb{V}_\infty}^{\oplus 2} \oplus \mathbb{R}^q$. Moreover, $\|A_n\|_{\mathrm{op}} = 1$, $\|B_n\|_{\mathrm{op}} \leq \max\{\sum_i |\lambda_i|, q\}$ for all n. Thus, both A_n and B_n extend to the continuous limit and Theorem 3.6 indeed applies and yields (36). ☐

The Schur-Horn orbitopes $\mathrm{SH}(x_1)_n$ are precisely the Schur-Horn orbitopes $\mathrm{SH}(\lambda)_n$ in (15) from Section 4.3 if $\mathbb{A}_n = \mathbb{R}^{m2^n \times m2^n}$ with $\mathsf{G}_n = \mathrm{O}(m2^n)$. They also reduce to the permutahedra $\mathrm{SH}(x_1)_n = \mathrm{Perm}(\lambda)_n$ as in (12) from Section 4.3 if $\mathbb{A}_n$ is the algebra of diagonal matrices in $\mathbb{M}_n$ with $\mathsf{G}_n = \mathrm{S}_{m2^n}$.

Proposition C.2 yields a generalization of the Schur-Horn theorem to AF algebras. Indeed, let $\mathbb{D}_n = \{x \in \mathbb{V}_n : x \text{ diagonal}\}$ which is a unital subalgebra of $\mathbb{A}_n$ embedded in $\mathbb{D}_{n+1}$ under φ_n. For $x_1 \in \mathbb{D}_1$, define

$$\mathrm{Perm}(x_1)_n = \mathrm{SH}(x_1)_n \cap \mathbb{D}_n.$$

The sequence of linear maps $(\mathrm{diag}_n\colon \mathbb{V}_n \to \mathbb{D}_n)_n$ extracting the diagonal of a Hermitian matrix in $\mathbb{V}_n$ extend to a bounded linear map $\mathrm{diag}\colon \overline{\mathbb{V}_\infty} \to \overline{\mathbb{D}_\infty}$ since $\mathrm{diag}_{n+1} \circ \varphi_n = \varphi_n \circ \mathrm{diag}_n$ and $\|\mathrm{diag}_n\|_{\mathrm{op}} = 1$. The finite-dimensional Schur-Horn theorem states that $\mathrm{diag}_n(\mathrm{SH}(x_1)_n) = \mathrm{Perm}(x_1)_n$. Using Proposition C.2, we obtain the following infinite-dimensional extension.

Proposition C.3. *Let $x_1 \in \mathbb{D}_1$ and let* $\mathrm{diag}\colon \overline{\mathbb{V}_\infty} \to \overline{\mathbb{D}_\infty}$ *be the bounded linear map extending the diagonal maps. Then* $\mathrm{diag}(\overline{\mathrm{SH}(x_1)_\infty}) = \overline{\mathrm{Perm}(x_1)_\infty}$.

Proof. Since $\overline{\mathrm{Perm}(x_1)_\infty} \subseteq \overline{\mathrm{SH}(x_1)_\infty}$ and diag is a projection onto $\overline{\mathbb{D}_\infty}$, we get $\mathrm{diag}(\overline{\mathrm{SH}(x_1)_\infty}) \supseteq \overline{\mathrm{Perm}(x_1)_\infty}$. Conversely, Proposition C.2 and continuity of diag yields

$$
\begin{aligned}
\mathrm{diag}(\overline{\mathrm{SH}(x_1)_\infty}) &\subseteq \overline{\left\{\sum\nolimits_{i=1}^{q} \lambda_i \mathrm{diag}(p_i) : p_i \in \overline{\mathcal{K}_\infty},\ \sum\nolimits_i p_i = 1,\ \tau(p_i) = m_i/m\right\}} \\
&= \overline{\left\{\sum\nolimits_{i=1}^{q} \lambda_i p_i : p_i \in \overline{\mathcal{K}_\infty} \cap \overline{\mathbb{D}_\infty},\ \sum\nolimits_i p_i = 1,\ \tau(p_i) = m_i/m\right\}} \\
&= \overline{\mathrm{Perm}(x_1)_\infty},
\end{aligned}
$$

where the last equality follows from Proposition C.2 applied to $\overline{\mathbb{D}_\infty}$ instead of $\overline{\mathbb{V}_\infty}$. □

The finite-dimensional Schur-Horn theorem yields $\mathrm{Perm}(x_1)_\infty = \mathrm{diag}(\mathrm{SH}(x_1)_\infty)$, and taking closures yields a weaker statement since it involves the closure of the image of diag. Proposition C.3 shows that this closure can be removed.

There is a considerable literature on extending the Schur-Horn theorem to infinite-dimensional setting, including the extension in [92] for operators with a finite spectrum, and in [93] for von Neumann algebras. As explained in [93, §1], removing the closure over the image of diag has been a major challenge in these more general settings. Our descriptions of limiting convex sets from Theorem 3.6 can be seen as resolving this challenge in our simpler setting of AF algebras.