



A note on convex relaxations for the inverse eigenvalue problem

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Abstract

The affine inverse eigenvalue problem consists of identifying a real symmetric matrix with a prescribed set of eigenvalues in an affine space. Due to its ubiquity in applications, various instances of the problem have been widely studied in the literature. Previous algorithmic solutions were typically nonconvex heuristics and were often developed in a case-by-case manner for specific structured affine spaces. In this short note we describe a general family of convex relaxations for the problem by reformulating it as a question of checking feasibility of a system of polynomial equations, and then leveraging tools from the optimization literature to obtain semidefinite programming relaxations. Our system of polynomial equations may be viewed as a matricial analog of polynomial reformulations of 0/1 combinatorial optimization problems, for which semidefinite relaxations have been extensively investigated. We illustrate numerically the utility of our approach in stylized examples that are drawn from various applications.

Keywords Combinatorial optimization · Real algebraic geometry · Schur–Horn orbitope · Semidefinite programming · Sums of squares polynomials

1 Introduction

The affine inverse eigenvalue problem (IEP) consists of identifying a real symmetric matrix with a prescribed set of eigenvalues in an affine space. IEPs arise in a range of applications in engineering and physical sciences, such as natural frequency identification in vibrating systems, pole placement, factor analysis, reliability testing, estimation of the Earth’s conductivity, graph partitioning and nuclear and molecular spectroscopy [3,6,7]. Further, there are many situations in which a question of interest is to solve a discrete inverse Sturm–Liouville problem [14], which is a special case of an affine IEP. Due to its ubiquity, IEPs have received much attention in the literature

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over the past several decades. On one end of the spectrum, there have been several efforts aimed at providing necessary and sufficient conditions for the existence of a solution of a given IEP [10,15,17]. For example, Landau proved that there always exists a symmetric Toeplitz matrix with a desired set of eigenvalues [17]; however, computing such matrices in a tractable manner remains a challenge. At the other end of the spectrum, several efforts have been aimed at developing efficient procedures for numerically finding solutions to particular types of the inverse eigenvalue problems [11,25], including some recent approaches based on convex optimization [19,26]. Our work differs from these approaches in two prominent ways. First, our framework is applicable to general affine IEPs, while some of the previous convex approaches are only useful for certain structured problem instances; see Sect. 3 for the broad range of examples to which we apply our methods. Second, we describe a family of convex relaxations for IEPs rather than just a single convex program, and our work allows for a tradeoff between computational cost and solution quality.

We begin by first reformulating the affine IEP as a question of checking the existence of a real solution to a system of polynomial equations. Formally, an instance of an affine IEP may be stated as follows:

Affine Inverse Eigenvalue Problem Given (i) a desired spectrum $\Lambda = \{(\lambda_i, m_i)\}_{i=1}^q \subset \mathbb{R} \times \mathbb{Z}_+$ of eigenvalue-multiplicity pairs with $\sum_i m_i = n$, and (ii) an affine space $\mathcal{E} = \{X \in \mathbb{S}^n : \text{Tr}(C_k X) = b_k, k = 1, \dots, \ell\}$, find an element of \mathcal{E} with spectrum given by Λ or certify that such a matrix does not exist. Here \mathbb{S}^n denotes the space of $n \times n$ real symmetric matrices.

This problem may be reformulated in terms of the existence of a real solution to the following system of equations:

$$S_{\text{iep}} := \begin{cases} f_1 := \sum_{i=1}^q Z_i - I = 0, \\ f_2^{(i)} := \text{Tr}(Z_i) - m_i = 0 \quad \text{for } i = 1, \dots, q, \\ f_3^{(i)} := Z_i^2 - Z_i = 0 \quad \text{for } i = 1, \dots, q, \\ f_4^{(k)} := \sum_{i=1}^q \lambda_i \text{Tr}(Z_i C_k) - b_k = 0 \quad \text{for } k = 1, \dots, \ell. \end{cases} \tag{1}$$

The variables in this system are the matrices $Z_1, \dots, Z_q \in \mathbb{S}^n$. The matrix I denotes the $n \times n$ identity. We are interested in whether the system of polynomials $S_{\text{iep}} = \{f_1, f_2^{(1)}, \dots, f_2^{(q)}, f_3^{(1)}, \dots, f_3^{(q)}, f_4^{(1)}, \dots, f_4^{(\ell)}\}$ has a common root over the reals, or in other words checking whether the associated real variety $\mathcal{V}_{\mathbb{R}}(S_{\text{iep}})$ is empty. To see that (1) is equivalent to an IEP instance, first we note that the constraints $\{f_3^{(i)}\}_{i=1}^q$ and $\{f_2^{(i)}\}_{i=1}^q$ ensure that the Z_i 's are projection matrices onto m_i dimensional subspaces. Second, by adding the constraint f_1 and noting that $\sum_i m_i = n$, we further ensure that the Z_i 's are matrices that project onto orthogonal subspaces whose direct sum equals \mathbb{R}^n . As such, the sum $\sum_i \lambda_i Z_i$ defines a matrix with spectrum specified by Λ . Finally, $f_4^{(k)}$ ensures that the resulting matrix $\sum_i \lambda_i Z_i$ lies in desired affine space \mathcal{E} .

The strength of this polynomial reformulation is that it allows us to leverage results from the optimization literature to systematically obtain convex relaxations for the affine IEP. Specifically, Parrilo [22] and Lasserre [18] developed hierarchies of semidefinite programming relaxations for polynomial optimization problems using

results from real algebraic geometry. These relaxations entail the solution of increasingly larger convex optimization problems that search over successively more complex collections of certificates that prove the infeasibility of the system defined by \mathcal{S}_{iep} . From a dual perspective, these relaxations may also be viewed as providing a sequence of convex outer approximations $\mathcal{R}_1(\Lambda, \mathcal{E}) \supseteq \mathcal{R}_2(\Lambda, \mathcal{E}) \supseteq \dots \supseteq \text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$, which leads to a natural heuristic for attempting to obtain solutions of the system \mathcal{S}_{iep} . We describe the mechanism to obtain these relaxations in Sect. 2. As an illustration, searching over a simple class of infeasibility certificates gives the following convex outer approximation to $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$:

$$\mathcal{R}_1(\Lambda, \mathcal{E}) = \left\{ (Z_1, \dots, Z_q) \in \otimes^q \mathbb{S}^n \mid \sum_{i=1}^q Z_i = I; \text{Tr}(Z_i) = m_i, Z_i \succeq 0 \forall i; \text{Tr}\left(\sum_{i=1}^q \lambda_i Z_i C_k\right) = b_k \forall k \right\}. \tag{2}$$

In Sect. 2.2 we relate the set $\mathcal{R}_1(\Lambda, \mathcal{E})$ to the *Schur–Horn orbitope* [23] associated to the spectrum Λ , which is the convex hull of all real symmetric matrices with spectrum Λ . If $\mathcal{R}_1(\Lambda, \mathcal{E}) = \emptyset$, then it is clear that $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}) = \emptyset$; otherwise, $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ may or may not be empty, and one can either attempt to find an element of $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ or search over a larger family of infeasibility certificates (see Sect. 2.1). In Sect. 2.3 we describe a convex outer approximation $\mathcal{R}_2(\Lambda, \mathcal{E})$ to $\text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$ that is in general tighter than $\mathcal{R}_1(\Lambda, \mathcal{E})$.

The description of $\mathcal{R}_1(\Lambda, \mathcal{E})$ in (2) consists of q semidefinite constraints on matrix variables of size $n \times n$. The description of $\mathcal{R}_2(\Lambda, \mathcal{E})$ in Sect. 2.3 involves a semidefinite constraint on a matrix variable of size $\binom{n+1}{2}q \times \binom{n+1}{2}q$. Tighter relaxations to $\text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$ than $\mathcal{R}_1(\Lambda, \mathcal{E})$ and $\mathcal{R}_2(\Lambda, \mathcal{E})$ require even larger semidefinite descriptions, and they become prohibitively expensive to solve for large n . Consequently, although we describe the general mechanism by which semidefinite relaxations of increasing size may be generated, we restrict our attention in numerical experiments to the performance of the relaxations $\mathcal{R}_1(\Lambda, \mathcal{E})$ and $\mathcal{R}_2(\Lambda, \mathcal{E})$. As the affine IEP includes (co-)NP-hard problems as special cases, these two relaxations generally do not solve every instance of an affine IEP (as expected); nonetheless, we demonstrate their effectiveness in Sect. 3 on stylized problems such as certifying non-existence of planted subgraphs, solving discrete Sturm–Liouville equations, and computing Toeplitz matrices with a desired spectrum.

Connection to combinatorial optimization A number of combinatorial problems such as computing the stability number of a graph or the knapsack problem may be formulated as checking feasibility of a system of equations in a collection of variables that take on values of 0/1. As many of these problems are NP-hard, a prominent approach to developing tractable approximations is to first specify the problems via polynomial equations and to then employ the methods referenced above to obtain semidefinite relaxations [2]. The polynomial reformulations consist of a system of equations defined by affine polynomials and quadratic equations of the form $x_i^2 - x_i = 0$ for each of the variables x_i to enforce the Boolean constraints. Our system (1) for the affine IEP may be viewed as a matricial analog of those arising in the literature on combinatorial problems, as the idempotence constraints $Z_i^2 - Z_i = 0$ represent a generalization of the scalar Boolean constraints $x_i^2 - x_i = 0$. The present note describes promising

experimental results of the performance of semidefinite relaxations for the affine IEP. As with the significant prior body of work on combinatorial optimization, it is of interest to investigate structural properties of our relaxations for specific affine spaces \mathcal{E} and spectra Λ . We outline future directions along these lines in Sect. 4.

Notation We use \otimes^q to denote the product space formed by q -tuples; i.e. $\otimes^q \mathbb{S}^n = \{(A_1, \dots, A_q) \mid A_1, \dots, A_q \in \mathbb{S}^n\}$. The usual convex hull operator is denoted by $\text{conv}(\cdot)$ and the usual trace operator is denoted by $\text{Tr}(\cdot)$.

2 Semidefinite relaxations for affine IEPs

2.1 From polynomial formulations to semidefinite relaxations

We summarize here the basic aspects of obtaining semidefinite relaxations for certifying infeasibility of a polynomial systems over the reals; we refer the reader to the survey [2] for further details. Let $\mathbb{R}[x]$ denote the ring of polynomials with real coefficients in indeterminates $x = (x_1, \dots, x_n)$. A *polynomial ideal* \mathcal{I} is a subset of $\mathbb{R}[x]$ that satisfies the following properties: (i) $0 \in \mathcal{I}$, (ii) $f_1, f_2 \in \mathcal{I} \Rightarrow f_1 + f_2 \in \mathcal{I}$, and (iii) $f \in \mathcal{I}, h \in \mathbb{R}[x] \Rightarrow hf \in \mathcal{I}$. The ideal generated by a collection of polynomials $f_1, \dots, f_t \in \mathbb{R}[x]$ is the set $\langle f_1, \dots, f_t \rangle = \{\sum_{i=1}^t f_i h_i \mid h_i \in \mathbb{R}[x]\}$ – here, f_1, \dots, f_t and h_1, \dots, h_t are referred to as *generators* and *coefficients*, respectively. The *real variety* corresponding to polynomials $g_1, \dots, g_r \in \mathbb{R}[x]$ is denoted $\mathcal{V}_{\mathbb{R}}(g_1, \dots, g_r) = \{x \in \mathbb{R}^n \mid g_i(x) = 0, i = 1, \dots, r\}$. Finally, the set of polynomials that can be expressed as a *sum of squares* of polynomials is denoted $\Sigma := \{p \in \mathbb{R}[x] \mid p = \sum_i p_i^2, p_i \in \mathbb{R}[x]\}$, where the sum is finite. We state next the *real Nullstellensatz* due to Krivine for certifying infeasibility of a system of a polynomial equations over \mathbb{R} :

Theorem 1 (Real Nullstellensatz) [16] *Given any collection of polynomials $f_1, \dots, f_t \in \mathbb{R}[x]$, we have that:*

$$-1 \in \Sigma + \langle f_1, \dots, f_t \rangle \iff \mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t) = \emptyset.$$

Here $-1 \in \mathbb{R}[x]$ refers to the constant polynomial. In general, the best-known bounds on the size of infeasibility certificates – i.e., the degrees of the polynomials in $\Sigma, \langle f_1, \dots, f_t \rangle$ that sum to -1 – are at least triply exponential. This is to be expected as many NP-hard problems (such as the induced subgraph isomorphism problem we discuss in Sect. 3) can be reformulated as certifying infeasibility of a system of polynomial equations.

Obtaining *tractable relaxations* based on the real Nullstellensatz relies on three key observations. First, one fixes a subset $\tilde{\mathcal{I}} \subset \langle f_1, \dots, f_t \rangle$ by considering polynomials $\sum_i h_i f_i$ in which the coefficients $h_i \in \mathbb{R}[x]$ have bounded degree (sets of the form $\tilde{\mathcal{I}}$ are sometimes called *truncated ideals*, although they are not formally ideals). In searching for infeasibility certificates of the form $-1 = p + q$, $p \in \Sigma, q \in \tilde{\mathcal{I}}$, one can check that without loss of generality the search for p can also be restricted to sum-of-squares polynomials of bounded degree; formally, if every element of $\tilde{\mathcal{I}}$ has

degree at most $2d$, one can restrict the search to elements of Σ with degree at most $2d$. Second, a decomposition $-1 = p + \sum_i h_i f_i$ where the p and the h_i 's all have bounded degree is a linear constraint in the coefficients of p and the h_i 's. Finally, checking that a polynomial $p \in \mathbb{R}[x]$ in n variables of degree at most $2d$ is an element of Σ can be formulated as a semidefinite feasibility problem; letting $m_{n,d}(x)$ denote the vector of all $\binom{n+d}{d}$ monomials in n variables of degree at most d , we have that:

$$p \in \Sigma \iff \exists P \in \mathbb{S}^{\binom{n+d}{d}}, P \succeq 0, p(x) = m_{n,d}(x)' P m_{n,d}(x).$$

The relation $p(x) = m_{n,d}(x)' P m_{n,d}(x)$ is equivalent to a set of linear equations relating the entries of P to the coefficients of p . Thus, the search over a restricted family of infeasibility certificates via bounding the degree of the coefficients of the elements of $\langle f_1, \dots, f_t \rangle$ is a semidefinite feasibility problem.

By considering a sequence of degree-bounded subsets $\mathcal{I}' \subset \mathcal{I}'' \subset \dots \subset \langle f_1, \dots, f_t \rangle$, one can search for more complex infeasibility certificates at the expense of solving increasingly larger semidefinite programs. Associated to this sequence of semidefinite programs is a sequence of *dual optimization problems* that provide successively tighter convex outer approximations to $\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t)$ (assuming strong duality holds), i.e., $\mathcal{R}' \supseteq \mathcal{R}'' \supseteq \dots \supseteq \text{conv}(\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t))$. This dual perspective is especially interesting for attempting to identify elements of $\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t)$. Concretely, fix a subset $\mathcal{I}' \subset \langle f_1, \dots, f_t \rangle$, and suppose that the search for an infeasibility certificate of the form $-1 \in \Sigma + \mathcal{I}'$ is unsuccessful. Then $\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t)$ may or may not be empty. At this stage, one can attempt to find an element of $\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t)$ by optimizing a random linear functional over the set \mathcal{R}' (obtained by considering the dual problem associated to the system $-1 \in \Sigma + \mathcal{I}'$), and checking whether the resulting optimal solution lies in $\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t)$; this heuristic is natural as $\mathcal{R}' \supseteq \text{conv}(\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t))$, and if these sets were equal then the heuristic would generically succeed at identifying an element of $\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t)$. If this approach to finding a solution is also unsuccessful, one can consider a larger subset $\mathcal{I}'' \subset \langle f_1, \dots, f_t \rangle$ and an associated tighter approximation $\mathcal{R}'' \supseteq \text{conv}(\mathcal{V}_{\mathbb{R}}(f_1, \dots, f_t))$ (here $\mathcal{I}' \subset \mathcal{I}''$ and $\mathcal{R}' \supseteq \mathcal{R}''$), and repeat the above procedure at a greater computational expense.

In Sects. 2.2 and 2.3, we employ the methodology described above to give concrete descriptions of two convex outer approximations of the variety specified by the system \mathcal{S}_{iep} associated to the affine IEP.

2.2 A first semidefinite relaxation

As our first example, we consider the following truncated ideal associated to the system \mathcal{S}_{iep} :

$$\begin{aligned} \mathcal{I}_1 = & \left\{ \text{Tr}(h_1 f_1) + \sum_{i=1}^q \left[h_2^{(i)} f_2^{(i)} + \text{Tr}\left(h_3^{(i)} f_3^{(i)}\right) \right] + \sum_{k=1}^{\ell} h_4^{(k)} f_4^{(k)} : h_2^{(i)}, h_4^{(k)} \text{ are degree-0 polynomials,} \right. \\ & \left. h_1, h_3^{(i)} \text{ are } n \times n \text{ symmetric matrices of degree-0 polynomials } \forall i, k \right\}. \end{aligned} \tag{3}$$

Here, each entry of the matrices h_1 and $h_3^{(i)}$ represents a degree zero real polynomial in entries of Z_i for $i = 1, \dots, q$ (i.e. a constant) that is the coefficient multiplying the corresponding index of f_1 and $f_3^{(i)}$, respectively. The truncated ideal $\mathcal{I}_1 \subset \langle \mathcal{S}_{\text{iep}} \rangle$ is obtained by restricting the coefficients to be constant polynomials (i.e., degree-zero polynomials). As a result, the elements of \mathcal{I}_1 consist of polynomials with degree at most two. Consequently, in searching for infeasibility certificates of the form $-1 \in \mathcal{I}_1 + \Sigma$ one need only consider quadratic polynomials in Σ , which in turn leads to checking feasibility of the following semidefinite program:

$$-\text{Tr}(A) - \sum_{i=1}^q m_i d_i - \sum_{k=1}^{\ell} b_k \xi_k = 1; \quad A + d_i I + \lambda_i \sum_{k=1}^{\ell} \xi_k C_k - B_{ii} = 0, \quad B_{ii} \geq 0, \quad i = 1, \dots, q \quad (4)$$

in variables $A \in \mathbb{S}^n$, $d_i \in \mathbb{R}$ and $B_{ii} \in \mathbb{S}^n$ for $i = 1, \dots, q$, and $\xi_k \in \mathbb{R}$ for $k = 1, \dots, \ell$. The elements of the truncated ideal \mathcal{I}_1 can be associated to the above problem via the relations $h_1 = -A$, $h_2^{(i)} = -d_i$, $h_3^{(i)} = -B_{ii}$, $h_4^{(k)} = -\xi_k$, and then observing that the constraints in (4) are equivalent to checking that the polynomial $\text{Tr}(h_1 f_1) + \sum_{i=1}^q h_2^{(i)} f_2^{(i)} + \sum_{i=1}^q \text{Tr}(h_3^{(i)} f_3^{(i)}) + \sum_{k=1}^{\ell} f_4^{(k)} h_4^{(k)}$ in variables (Z_1, \dots, Z_q) can be decomposed as $-1 - \Sigma$ (this can be observed by substituting the coefficients in the polynomial and grouping the terms that multiply Z_i together for $i = 1, \dots, q$). Next we relate $\mathcal{R}_1(\Lambda, \mathcal{E})$ and the system $-1 \in \mathcal{I}_1 + \Sigma$ via strong duality:

Proposition 1 Consider an affine IEP specified by a spectrum Λ and an affine space $\mathcal{E} \subset \mathbb{S}^n$. Let \mathcal{I}_1 be defined as in (3) and $\mathcal{R}_1(\Lambda, \mathcal{E})$ as in (2). Then exactly one of the following two statements is true:

- (1) $\mathcal{R}_1(\Lambda, \mathcal{E})$ is nonempty,
- (2) $-1 \in \mathcal{I}_1 + \Sigma$.

Proof Here, the feasibility of the system (4) is equivalent to the condition $-1 \in \mathcal{I}_1 + \Sigma$. One can check that the system (4) and the constraints describing $\mathcal{R}_1(\Lambda, \mathcal{E})$ are strong alternatives of each other, which follows from an application of conic duality – strong duality follows from Slater’s condition being satisfied.

Proof We begin by posing a non-convex feasibility problem with objective function of minimizing 0 and constraints given by the system of equations \mathcal{S}_{iep} . The dual problem of this system is equivalent to the following convex program:

$$\begin{aligned} \max_{\substack{A, B_{ii} \in \mathbb{S}^n; d_i, \xi_k \in \mathbb{R} \\ i=1, \dots, q; k=1, \dots, \ell}} & -\text{Tr}(A) - \sum_{i=1}^q m_i d_i - \sum_{k=1}^{\ell} b_k \xi_k \\ \text{s.t.} & A + d_i I + \lambda_i \sum_{k=1}^{\ell} \xi_k C_k - B_{ii} = 0, \quad B_{ii} \geq 0, \quad i = 1, \dots, q \end{aligned} \quad (5)$$

Note that convex program (5) is homogeneous, and thus, if equation (4) holds, then optimal value of (5) is infinity, which, by weak duality proves that \mathcal{S}_{iep} is infeasible. Therefore, the feasibility of the system (4) is equivalent to the condition $-1 \in \mathcal{I}_1 + \Sigma$.

Next, we take the the dual of system (5) to obtain a convex outer-approximation to the solution set of \mathcal{S}_{iep} . The dual of the system (5) is equivalent to optimizing over $\mathcal{R}_1(\Lambda, \mathcal{E})$ with a vacuous objective of minimizing 0. Here, the Slater’s condition holds, and as such, the system (5) and its dual have the same optimal value. Therefore, whenever $\mathcal{R}_1(\Lambda, \mathcal{E})$ is nonempty, both convex program (5) and its dual have optimal value 0, system (4) is infeasible and hence $-1 \notin \mathcal{I}_1 + \Sigma$. In contrast, if $\mathcal{R}_1(\Lambda, \mathcal{E})$ is empty, then both (5) and its dual have optimal value $+\infty$, system (4) is feasible and thus $-1 \in \mathcal{I}_1 + \Sigma$.

As a consequence of this result, it follows that $\mathcal{R}_1(\Lambda, \mathcal{E})$ is a convex outer approximation of the variety $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$. A more direct way to see this is to consider any element $(Z_1, \dots, Z_q) \in \mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ and to note that the idempotence constraints $Z_i^2 - Z_i = 0$ in \mathcal{S}_{iep} imply the semidefinite constraints $Z_i \geq 0$ in $\mathcal{R}_1(\Lambda, \mathcal{E})$.

The set $\mathcal{R}_1(\Lambda, \mathcal{E})$ is closely related to the Schur–Horn orbitope associated to the spectrum Λ [23]:

$$\begin{aligned} \text{SH}(\Lambda) &= \text{conv}\{M \in \mathbb{S}^n \mid \lambda(M) = \Lambda\} \\ &= \left\{ X \in \mathbb{S}^n \mid \exists (Z_1, \dots, Z_q) \in \otimes^q \mathbb{S}^n \text{ s.t. } \sum_{i=1}^q Z_i = I; \text{Tr}(Z_i) = m_i, Z_i \geq 0 \forall i; X = \sum_{i=1}^q \lambda_i Z_i \right\}. \end{aligned} \tag{6}$$

The second equality follows from the characterization in [9]. The Schur–Horn orbitope was so-named by the authors of [23] due to its connection with the Schur–Horn theorem. A subset of the authors of the present note employed the Schur–Horn orbitope in developing efficient convex relaxations for NP-hard combinatorial optimization problems such as finding planted subgraphs [4] and computing edit distances between pairs of graphs [5]. In the context of the present note, the Schur–Horn orbitope provides a precise characterization of the conditions under which $-1 \in \mathcal{I}_1 + \Sigma$ is successful. Specifically, from Proposition 1 and (6) we have that:

$$-1 \in \mathcal{I}_1 + \Sigma \iff \mathcal{R}_1(\Lambda, \mathcal{E}) = \emptyset \iff \text{SH}(\Lambda) \cap \mathcal{E} = \emptyset. \tag{7}$$

Hence, if $-1 \notin \mathcal{I}_1 + \Sigma$, we have that $\mathcal{R}_1(\Lambda, \mathcal{E}) \neq \emptyset$. In particular, the variety $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ may or may not be empty. At this stage, as discussed in Sect. 2.1 one can maximize a random linear functional over the set $\mathcal{R}_1(\Lambda, \mathcal{E})$; the resulting optimal solution $(\hat{Z}_1, \dots, \hat{Z}_q)$ is generically an extreme point of $\mathcal{R}_1(\Lambda, \mathcal{E})$, and one can check if $(\hat{Z}_1, \dots, \hat{Z}_q)$ satisfies the equations in the system \mathcal{S}_{iep} . If this attempt at finding a feasible point in $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ is unsuccessful, one can repeat the preceding steps at attempting to certify infeasibility or to find a feasible point in $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ via a larger semidefinite program, which we describe in the next subsection.

2.3 A tighter semidefinite relaxation

Next we consider a larger truncated ideal $\mathcal{I}_2 \subset \langle \mathcal{S}_{\text{iep}} \rangle$ with larger degree coefficients than in \mathcal{I}_1 :

$$\mathcal{I}_2 = \left\{ \text{Tr} (h_1 (Z_1, \dots, Z_q) f_1) + \sum_{i=1}^q h_2^{(i)} (Z_1, \dots, Z_q) f_2^{(i)} + \sum_{i=1}^q \text{Tr} (h_3^{(i)} f_3^{(i)}) + \sum_{k=1}^l f_4^{(k)} h_4^{(k)} (Z_1, \dots, Z_q) : \right.$$

$h_1 (Z_1, \dots, Z_q)$ is an $n \times n$ matrix whose entries are affine polynomials,
 $h_3^{(i)}$ are $n \times n$ matrices whose entries are degree 0 polynomials,
 $h_2^{(i)} (Z_1, \dots, Z_q), h_4^{(k)} (Z_1, \dots, Z_q)$ are affine polynomials $\forall i = 1, \dots, q, k = 1, \dots, \ell$. (8)

As before, all polynomials referred to in the definition of \mathcal{I}_2 are polynomials in the entries of $Z_i, i = 1, \dots, q$. The coefficients $h_3^{(i)}$ are constrained in the same way as in \mathcal{I}_1 but the other coefficients $h_1, h_2^{(i)}, h_4^{(k)}$ are allowed to be affine polynomials (in the case of h_1 , more precisely a matrix of affine polynomials). Here, we utilize this particular choice of coefficient degrees in order to ensure that the resulting collection \mathcal{I}_2 consists of polynomials of degree at most two, and therefore we can restrict our attention to elements of Σ of degree at most two in searching for infeasibility certificates of the form $-1 \in \mathcal{I}_2 + \Sigma$. However, \mathcal{I}_2 is in general larger than \mathcal{I}_1 so that $\mathcal{I}_2 + \Sigma$ offers a richer family of infeasibility certificates than $\mathcal{I}_1 + \Sigma$. The convex relaxation $\mathcal{R}_2(\Lambda, \mathcal{E})$ obtained as an alternative to the system $-1 \in \mathcal{I}_2 + \Sigma$ in turn provides a tighter approximation in general than $\mathcal{R}_1(\Lambda, \mathcal{E})$ to the convex hull $\text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$. We require some notation to give a precise description of $\mathcal{R}_2(\Lambda, \mathcal{E})$. Let $\delta_{k,l}$ denote the usual delta function which equals one if the arguments are equal and zero otherwise. Additionally, for $s, t = 1, \dots, n$ let

$$f_{s,t} = \begin{cases} e_s e_t^T, & \text{if } s = t, \\ \frac{1}{2}(e_s e_t^T + e_t e_s^T), & \text{otherwise.} \end{cases}$$

Here, $e_s, e_t \in \mathbb{R}^n$ are the s 'th and t 'th standard basis vectors in \mathbb{R}^n . The set $\mathcal{R}_2(\Lambda, \mathcal{E})$ is then specified as:

$$\begin{aligned}
 \mathcal{R}_2(\Lambda, \mathcal{E}) = & \left\{ (Z_1, \dots, Z_q) \in \otimes^q \mathbb{S}^n \mid \exists \mathcal{W}_{i,j} : \mathbb{S}^n \rightarrow \mathbb{S}^n, i, j = 1, \dots, q, \exists \mathfrak{W} : \otimes^q \mathbb{S}^n \rightarrow \otimes^q \mathbb{S}^n, \right. \\
 & \mathfrak{W} \succeq 0, [\mathfrak{W}(X_1, \dots, X_q)]_i = \sum_{j=1}^q \mathcal{W}_{i,j}(X_j) \quad i = 1, \dots, q, Z_i \succeq 0 \quad i = 1, \dots, q \\
 & \sum_{i=1}^q Z_i = I, \operatorname{Tr}(Z_i) = m_i, \quad i = 1, \dots, q, \sum_{i=1}^q \lambda_i \operatorname{Tr}(Z_i C_k) = b_k, \quad k = 1, \dots, l, \\
 & \sum_{j=1}^q \mathcal{W}_{i,j}(f_{s,t}) = \delta_{s,t} Z_i, \quad s, t = 1, \dots, n, \quad i = 1, \dots, q, \\
 & \sum_{s=1}^n \mathcal{W}_{i,j}(f_{s,s}) = m_j Z_i, \quad i, j = 1, \dots, q, \\
 & \sum_{r=1}^n \operatorname{Tr}(f_{s,r} \mathcal{W}_{i,i}(f_{t,r})) = (Z_i)_{s,t}, \quad i = 1, \dots, q, \quad s, t = 1, \dots, n, \\
 & \left. \sum_{j=1}^q \lambda_j \mathcal{W}_{i,j}(C_k) = b_k Z_i, \quad i = 1, \dots, q, \quad k = 1, \dots, \ell \right\}.
 \end{aligned} \tag{9}$$

Our next result records the fact that $\mathcal{R}_2(\Lambda, \mathcal{E})$ does constitute a strong alternative for $-1 \in \mathcal{I}_2 + \Sigma$.

Proposition 2 *Consider an affine IEP specified by a spectrum Λ and an affine space $\mathcal{E} \subset \mathbb{S}^n$. Let \mathcal{I}_2 be defined as in (8) and $\mathcal{R}_2(\Lambda, \mathcal{E})$ as in (9). Then exactly one of the following two statements is true:*

- (1) $\mathcal{R}_2(\Lambda, \mathcal{E})$ is nonempty,
- (2) $-1 \in \mathcal{I}_2 + \Sigma$.

Proof The proof is almost identical to that of Proposition 1, and it follows from an application of conic duality. The central difference is that when taking the first dual, we utilize higher degree dual variables to incorporate the higher degree terms in \mathcal{I}_2 .

It is clear that $\mathcal{R}_1(\Lambda, \mathcal{E}) \supseteq \mathcal{R}_2(\Lambda, \mathcal{E})$ as the constraints defining $\mathcal{R}_2(\Lambda, \mathcal{E})$ are a superset of those defining $\mathcal{R}_1(\Lambda, \mathcal{E})$. Further, for any $(Z_1, \dots, Z_q) \in \mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$, one can check that the constraints defining $\mathcal{R}_2(\Lambda, \mathcal{E})$ are satisfied by setting the linear operators $\mathcal{W}_{i,j}(X) = \operatorname{Tr}(Z_j X) Z_i \forall X \in \mathbb{S}^n$. Thus, there are additional *quadratic relations* among the Z_i 's that are satisfied by the elements of $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ and are implied by $\mathcal{R}_2(\Lambda, \mathcal{E})$, but are not captured by the set $\mathcal{R}_1(\Lambda, \mathcal{E})$. This is the source of the improvement of the relaxation $\mathcal{R}_2(\Lambda, \mathcal{E})$ compared to $\mathcal{R}_1(\Lambda, \mathcal{E})$, although the improvement comes at the expense of solving a substantially larger semidefinite program. In particular, $\mathcal{R}_1(\Lambda, \mathcal{E})$ entails checking q semidefinite constraints on $n \times n$ real symmetric matrices, while the description of $\mathcal{R}_2(\Lambda, \mathcal{E})$ involves a semidefinite constraint on the operator $\mathfrak{W} : \otimes^q \mathbb{S}^n \rightarrow \otimes^q \mathbb{S}^n$, which is equivalent to stipulating that a $\binom{n+1}{2} q \times \binom{n+1}{2} q$ real symmetric matrix is positive semidefinite. Thus, optimizing over $\mathcal{R}_2(\Lambda, \mathcal{E})$ is much more computationally expensive than $\mathcal{R}_1(\Lambda, \mathcal{E})$. In general, optimizing over $\mathcal{R}_2(\Lambda, \mathcal{E})$ for relatively large values of n and q might require employing specialized solvers that exploits problem structure rather than employing general-purpose solvers.

3 Numerical illustrations

Here we present experiments illustrating the performance of the relaxations $\mathcal{R}_1(\Lambda, \mathcal{E})$, $\mathcal{R}_2(\Lambda, \mathcal{E})$ on random problem instances and stylized instances arising in applications. Our results are obtained using the CVX parser [13] and the SDPT3 solver [24]. Before describing these, we present an approach to strengthen the relaxation $\mathcal{R}_2(\Lambda, \mathcal{E})$ by adding valid constraints without increasing the size of the semidefinite inequality.

3.1 Strengthening the relaxations

A prominent approach in the optimization literature for obtaining improved bounds on hard nonconvex problems is to add redundant constraints. The procedure presented in Sect. 2.1 of considering a sequence of truncated ideals $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \dots \subseteq \langle \mathcal{S}_{\text{iep}} \rangle$ is a systematic method to add valid constraints; in particular, the elements of \mathcal{I}_1 and \mathcal{I}_2 represent polynomials that vanish at all the points in $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$. As $\mathcal{I}_1 \subseteq \mathcal{I}_2$, the relaxation $\mathcal{R}_2(\Lambda, \mathcal{E})$ offers (in general) a tighter convex outer approximation of $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ than $\mathcal{R}_1(\Lambda, \mathcal{E})$ as $\mathcal{R}_2(\Lambda, \mathcal{E})$ is derived from the incorporation of a larger collection of redundant constraints.

Here we present a simple alternative approach to adding redundant constraints for the affine IEP by augmenting the original system \mathcal{S}_{iep} with additional polynomials that vanish on $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$, and which are not contained in the truncated ideals $\mathcal{I}_1, \mathcal{I}_2$. Specifically, we consider the following modified system of equations:

$$\mathcal{S}_{\text{iep}}^+ = \mathcal{S}_{\text{iep}} \cup \{Z_i Z_j, i, j = 1, \dots, q, i \neq j\}.$$

The matrix equations $Z_i Z_j = 0$ are satisfied for $i \neq j$ by every solution of \mathcal{S}_{iep} as a consequence of the vanishing of $f_1, f_2^{(i)}, f_3^{(i)}$. However, despite being of low degree, the matrix polynomials $Z_i Z_j, i \neq j$ are not contained in $\mathcal{I}_1, \mathcal{I}_2$. Consequently, incorporating these degree-two equations offers the prospect of strengthening our relaxations without a significant additional computational expense. We define truncated ideals $\mathcal{I}_1^+, \mathcal{I}_2^+$ corresponding to $\mathcal{S}_{\text{iep}}^+$ in an identical fashion to $\mathcal{I}_1, \mathcal{I}_2$ by restricting the coefficients corresponding to the additional polynomials $Z_i Z_j, i \neq j$ to be matrices of constant polynomials (as in the restriction of the coefficients $h_3^{(i)}$ of $f_3^{(i)}$), with the coefficients of the other polynomials $f_1, f_2^{(i)}, f_3^{(i)}, f_4^{(k)}$ being as in $\mathcal{I}_1, \mathcal{I}_2$.

The semidefinite relaxation $\mathcal{R}_1^+(\Lambda, \mathcal{E})$ obtained as a strong alternative to the system $-1 \in \mathcal{I}_1^+ + \Sigma$ is identical to $\mathcal{R}_1(\Lambda, \mathcal{E})$, i.e., the additional redundant constraints do not strengthen the relaxation. However, the strong alternative to the system $-1 \in \mathcal{I}_2^+ + \Sigma$ leads to a convex outer approximation $\mathcal{R}_2^+(\Lambda, \mathcal{E})$ of $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ that is in general tighter than $\mathcal{R}_2(\Lambda, \mathcal{E})$; in addition to all the constraints that define $\mathcal{R}_2(\Lambda, \mathcal{E})$ in (9), the set $\mathcal{R}_2^+(\Lambda, \mathcal{E})$ consists of the additional constraints $\sum_{r=1}^n \text{Tr}(f_{s,r} \mathcal{W}_{i,j}(f_{t,r})) = 0, i, j = 1, \dots, q, i \neq j, s, t = 1, \dots, n$ on the variables $\mathcal{W}_{i,j}$. Thus, a notable feature of the relaxation $\mathcal{R}_2^+(\Lambda, \mathcal{E})$ is that it is of the same size as $\mathcal{R}_2(\Lambda, \mathcal{E})$, despite providing a tighter convex outer approximation in general to $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$. We demonstrate the merits of this relaxation in the numerical experiments in this section.

3.2 Experiments with random affine IEPs

We present an experiment on random problems instances in this subsection. Specifically, we compare the relative power of the two relaxations described in Sect. 2 in certifying infeasibility, or from a dual viewpoint, in approximating $\text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$. To provide a visual illustration, we consider affine IEPs involving matrices in \mathbb{S}^3 , with a desired spectrum of $\{-1, 0, 1\}$. We begin by considering an affine space defined by $\ell = 3$ random linear equations, i.e., $\mathcal{E} = \{X \in \mathbb{S}^3 \mid \text{Tr}(C_k X) = 0, C_k \in \mathbb{S}^3, k = 1, \dots, \ell\}$, where the C_k 's have random entries. Given the spectrum (which fixes the trace) and the affine space \mathcal{E} , the solution set $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ is constrained to lie in an affine space of dimension at most two in \mathbb{S}^3 . We then set the entries X_{11}, X_{22} to fixed values in the range $[-1, 1]$, and check whether there exists a matrix in \mathbb{S}^3 with these values for X_{11}, X_{22} that can be expressed as $\sum_i \lambda_i Z_i$ for $(Z_1, Z_2, Z_3) \in \mathcal{R}_1(\Lambda, \mathcal{E})$ and for $(Z_1, Z_2, Z_3) \in \mathcal{R}_2(\Lambda, \mathcal{E})$. Fig. 1a, b represent two different problem instances obtained by generating two affine spaces \mathcal{E} as described above, and they illustrate graphically when the relaxations succeed or fail at certifying infeasibility. Evidently, the relaxation $\mathcal{R}_2(\Lambda, \mathcal{E})$ is successful in certifying infeasibility over a larger range of values of X_{11}, X_{22} than the relaxation $\mathcal{R}_1(\Lambda, \mathcal{E})$, thus illustrating its increased power (at a greater computational expense). From a dual perspective, we have in both cases that $\mathcal{R}_1(\Lambda, \mathcal{E}) \supsetneq \mathcal{R}_2(\Lambda, \mathcal{E})$. In particular, the feasibility regions corresponding to $\mathcal{R}_1(\Lambda, \mathcal{E})$ and $\mathcal{R}_2(\Lambda, \mathcal{E})$ in Fig. 1a, b represent the projections of these sets onto the (X_{11}, X_{22}) -plane of \mathbb{S}^3 . In each of the two settings, we maximized 1000 random linear functionals over $\mathcal{R}_2(\Lambda, \mathcal{E})$ and in all cases obtained an element of $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$. Consequently, it appears at least based on numerical evidence that $\mathcal{R}_2(\Lambda, \mathcal{E}) = \text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$ in both examples. Figure 1c, d give two examples based on the same setup as above, but with $\ell = 2$ random linear equations defining the affine space \mathcal{E} . Here the dimension of the solution set $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ is at most three, and the feasibility regions corresponding to $\mathcal{R}_1(\Lambda, \mathcal{E})$ and $\mathcal{R}_2(\Lambda, \mathcal{E})$ in Fig. 1c, d represent two-dimensional projections (onto the (X_{11}, X_{22}) -plane of \mathbb{S}^3) of these sets. As with the previous examples, we maximized 1000 random linear functionals over $\mathcal{R}_2(\Lambda, \mathcal{E})$ and in all cases obtained an element of $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$. Consequently, it again appears that $\mathcal{R}_2(\Lambda, \mathcal{E}) = \text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$.

3.3 Discrete inverse Sturm–Liouville problem

Next, we demonstrate an application of our framework to certify infeasibility of, or produce a solution to, the extensively studied discrete inverse Sturm–Liouville problem [14]. This problem arises as a discretization of a continuous differential boundary problem of the form $-u''(x) + p(x)u(x) = \lambda u(x)$, $u(0) = u(\pi) = 0$. Here, $u(x)$ and $p(x)$ are functions, and λ is a parameter that is an eigenvalue of the system. A particular discretization of this differential equation gives rise to the linear system $\left(\frac{(n+1)^2}{\pi^2} J + D\right) u = \lambda u$, where J is a Jacobian matrix with diagonal entries equal to 2 and the nonzero off-diagonal entries equal to -1 [14]. Hence, given a collection $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, one wishes to identify a diagonal matrix D so that this linear system

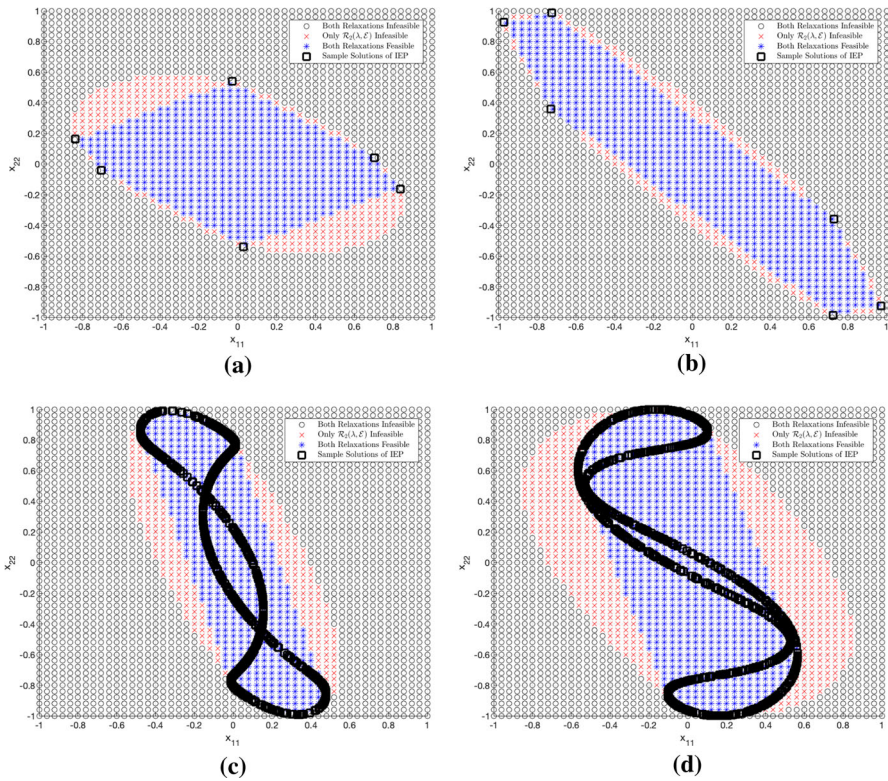


Fig. 1 Comparison of feasible/infeasible regions of $\mathcal{R}_1(\Lambda, \mathcal{E})$ and $\mathcal{R}_2(\Lambda, \mathcal{E})$ for four random problem instances as described in Sect. 3.2. The points marked with black circles, red crosses, and blue stars correspond, respectively, to settings in which $\mathcal{R}_1(\Lambda, \mathcal{E})$ and $\mathcal{R}_2(\Lambda, \mathcal{E})$ are both infeasible; $\mathcal{R}_1(\Lambda, \mathcal{E})$ is feasible and $\mathcal{R}_2(\Lambda, \mathcal{E})$ is infeasible; and both $\mathcal{R}_1(\Lambda, \mathcal{E})$ and $\mathcal{R}_2(\Lambda, \mathcal{E})$ are feasible. Thick black squares represent (X_{11}, X_{22}) values of solutions to the affine IEP

has a solution for each setting $\lambda = \lambda_i$, i.e., $\lambda_1, \dots, \lambda_n$ are eigenvalues of the matrix $\frac{(n+1)^2}{\pi^2} J + D$. This is clearly an instance of an affine IEP.

We consider two different instantiations of the problem with $n = 5$. First, we consider the set of eigenvalues $\{1, 2, 3, 4, 5\}$. In this instance, there exists a decomposition $-1 \in \mathcal{I}_1 + \Sigma$ which certifies that the discrete inverse Sturm–Liouville problem is infeasible with the given spectrum. Next, we consider eigenvalues in the set $\{1, 4, 9, 16, 25\}$. In this case, the discrete inverse Sturm–Liouville problem turns out to be feasible. Specifically, we attempt to produce a solution to the inverse discrete Sturm–Liouville problem by maximizing 100 random linear functionals over the convex sets $\mathcal{R}_1(\Lambda, \mathcal{E})$, $\mathcal{R}_2(\Lambda, \mathcal{E})$, and $\mathcal{R}_2^+(\Lambda, \mathcal{E})$; our approach succeeds 14 out of 100 times over $\mathcal{R}_1(\Lambda, \mathcal{E})$, 26 out of 100 times over $\mathcal{R}_2(\Lambda, \mathcal{E})$, and 55 out of 100 times over $\mathcal{R}_2^+(\Lambda, \mathcal{E})$. These results suggest that our semidefinite relaxations may offer a useful solution framework across the range of applications in which the discrete inverse Sturm–Liouville problem arises.

3.4 Induced subgraph isomorphism

We present next the utility of our framework in the context of a problem in combinatorial optimization, namely the induced subgraph isomorphism problem. Here we are given two undirected, unweighted graphs \mathcal{G} and \mathcal{G}' on n and n' vertices, respectively, with $n' < n$. The problem is to determine whether \mathcal{G}' is an induced subgraph of \mathcal{G} . This problem is NP-complete in general and has received considerable attention.

Suppose \mathcal{G}' is an induced subgraph of \mathcal{G} . Letting $A \in \mathbb{S}^n$ and $A' \in \mathbb{S}^{n'}$ be adjacency matrices representing the graphs \mathcal{G} and \mathcal{G}' , respectively, such that A' is equal to a principal submatrix of A , there must exist a matrix $M \in \mathbb{S}^n$ that satisfies the following conditions:

$$\text{Tr}(AM) = \sum_{i,j=1}^{n'} (A')_{i,j}; \quad (M)_{i,j} = 0 \text{ if } (A)_{i,j} = 0, \quad i, j = 1, \dots, n. \quad (10)$$

This consequence follows because we may choose M to be equal to A' on the $n' \times n'$ principal submatrix corresponding to corresponding to \mathcal{G}' and zero elsewhere. Thus, a sufficient condition to certify that \mathcal{G}' is not an induced subgraph of \mathcal{G} is to certifying the infeasibility of an affine IEP in which the spectrum is equal to that of A' along with an eigenvalue of zero with multiplicity $n - n'$ and the affine space is given by (10).

With this approach, we prove that the octahedral graph with 6 nodes and 12 edges (shown in Fig. 2a) is not contained as an induced subgraph in either of the larger graphs shown in Fig. 2b (on 20 nodes with 44 edges) and Fig. 2c (on 15 nodes with 38 edges). Both of these larger graphs are randomly generated Erdős–Renyi random graphs where any two vertices are independently and randomly connected with probability 0.2 for Fig. 2b and 0.4 for Fig. 2c. For the first graph, there exists a decomposition $-1 \in \mathcal{I}_1 + \Sigma$, thus certifying that the octahedral graph is not an induced subgraph. For the second graph, there is no infeasibility certificate of the form $-1 \in \mathcal{I}_1 + \Sigma$ but there is one of the form $-1 \in \mathcal{I}_2^+ + \Sigma$, thus providing a certificate that the octahedral graph is again not an induced subgraph.

3.5 Constructing a real symmetric Toeplitz matrix with desired spectrum

Finally, we describe how our framework can be utilized for constructing real symmetric Toeplitz matrices with a desired spectrum. As Toeplitz matrices form a subspace, this question is an instance of an affine IEP. Landau showed that there exists a Toeplitz matrix with a desired spectrum, but his proof was non-constructive [17], and numerically constructing such matrices continues to remain a challenge.

In our first experiment, we set $n = 5$ and consider the problem of constructing a symmetric Toeplitz matrix with eigenvalues $\{1, 2, 3, 4, 5\}$. We maximize random linear functionals over the sets $\mathcal{R}_1(\Lambda, \mathcal{E})$, $\mathcal{R}_2(\Lambda, \mathcal{E})$ and $\mathcal{R}_2^+(\Lambda, \mathcal{E})$, and we succeed at identifying a Toeplitz matrix with the desired spectrum 4 out of 100 times with $\mathcal{R}_1(\Lambda, \mathcal{E})$, 12 out of 100 times with $\mathcal{R}_2(\Lambda, \mathcal{E})$, and 41 out of 100 times with $\mathcal{R}_2^+(\Lambda, \mathcal{E})$. In our second experiment we set $n = 8$ and we seek a Toeplitz matrix with eigenvalues

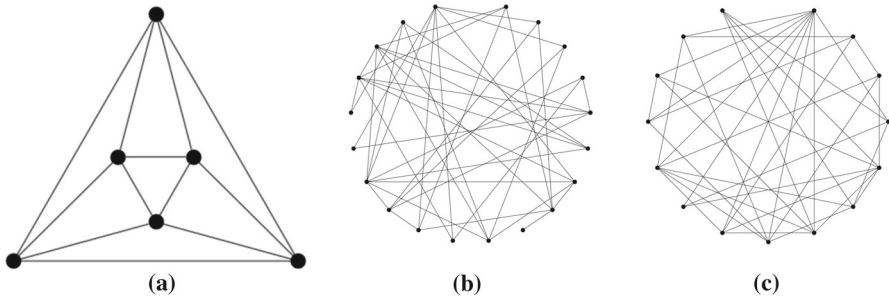


Fig. 2 From left to right: the octahedral graph, an Erdős–Rényi random graph on 20 nodes with $p = 0.2$, an Erdős–Rényi random graph on 15 nodes with $p = 0.4$. Our first convex relaxation certifies that the octahedral graph is not an induced subgraph of the graph shown in (b). A tighter convex relaxation proves the same result for the graph shown in (c)

-1 (with multiplicity four) and 1 (with multiplicity four). With the same approach as before of maximizing random linear functionals, we identify a Toeplitz matrix with the desired spectrum 17 out of 100 times with $\mathcal{R}_1(\Lambda, \mathcal{E})$, and 84 out of 100 times with both $\mathcal{R}_2(\Lambda, \mathcal{E})$ and $\mathcal{R}_2^+(\Lambda, \mathcal{E})$. In summary, our framework provides a numerical counterpart to Landau’s non-constructive existence result.

4 Conclusions

In this short note we describe a new framework for the affine IEP by first formulating it as a system of polynomial equations and then employing techniques from the polynomial optimization literature to obtain several semidefinite relaxations. These relaxations offer increasingly tighter approximations at the expense of solving larger semidefinite programming problems. We compare these relaxations both in random problem instances as well as in stylized examples in the context of various applications.

A number of future directions arise from our work. First, it is of practical interest to characterize conditions – stated in terms of the spectrum Λ and the affine space \mathcal{E} – under which a particular relaxation such as $\mathcal{R}_1(\Lambda, \mathcal{E})$ is tight; that is, $\mathcal{R}_1(\Lambda, \mathcal{E}) = \text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$. Such characterizations provide families of affine IEP instances that are exactly solved by the first level of our semidefinite relaxation hierarchy. Second, a closely related question is whether the hierarchy of convex relaxations is always tight at a *finite level*; that is, does equality $\mathcal{R}_k(\Lambda, \mathcal{E}) = \text{conv}(\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}}))$ always hold for some *finite* k . Such questions are referred to as *finite convergence* in the literature. There are a number of prior works that aim to provide sufficient conditions under which finite convergence occurs [12,21]; in our context for example, it is known that finite convergence occurs when the set of solutions to $\mathcal{V}_{\mathbb{R}}(\mathcal{S}_{\text{iep}})$ is finite [12]. Third, it would be of interest to develop methods for extracting solutions to an inverse eigenvalue problem instance beyond our proposed approach based on maximizing random linear functionals described in Sect. 2.1. For instance, in the context of polynomial optimization, the notion of a *flat truncation* [20] is useful for certifying when a certain level of the convex relaxation obtained via the SOS hierarchy is tight,

and for subsequently extracting minimizers of the polynomial optimization instance. It would be interesting to specialize these ideas to our setup. Fourth, a future direction of practical interest is to develop scalable methods for solving higher levels of our hierarchy beyond the first level. A recent development in this space proposes the use of linear programming and second order cone programming certificates of polynomial non-negativity for solving polynomial optimization instances [1,8]. These techniques are applicable in our setup. Since these techniques rely on families of certificates that are a strict subset of sums-of-squares, but are computationally cheaper to search over, it would be interesting to understand the associated computational tradeoffs.

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