CS 38 Introduction to Algorithms **Spring 2014**

Problem Set 6

Out: May 20 *Due:* **May 27**

Reminder: you are encouraged to work in groups of two or three; however you must turn in your own write-up and note with whom you worked. You may consult the course notes and the optional text (CLRS). The full honor code guidelines can be found in the course syllabus.

Please attempt all problems. **To facilitate grading, please turn in each problem on a separate sheet of paper and put your name on each sheet. Do not staple the separate sheets.**

1. Show that problem of finding the maximum cardinality independent set in the intersection of *three* matroids is NP-hard. That is, specify three matroids over a common universe E (prove that each is a matroid), together with a polynomial-time procedure for each one that determines whether or not a set $A \subseteq E$ is an independent set. Then give a reduction from an NP-complete problem to the problem of determining whether there exists an independent set of cardinality at least k in the intersection of the three matroids. Hint: reduce from Hamilton path in a directed graph.

2. Kernalization. Given a graph $G = (V, E)$ and a parameter k, we are interested in whether G contains a vertex cover of size k. Give a polynomial time procedure that produces, from G and k, an integer $k' \leq k$ and a graph $G' = (V', E')$ with $|E| \leq k^2$ and $|V'| \leq 2k^2$ such that G' has a vertex cover of size k' iff G has a vertex cover of size k. This results in a $O(k^{2k^2} + \text{poly}(n))$ time procedure to solve vertex cover with parameter k. Hint: consider the following two operations: (1) remove a vertex of degree greater than k and all of its incident edges, and (2) remove an isolated (degree 0) vertex.

3. We are given a directed graph $G = (V, E)$, representing a communication network, together with a source vertex s and a sink vertex t. Let P be the set of s-t paths in G. We are transmitting information along each path $p \in P$, and each path utilizes a $0 \leq x_p \leq 1$ fraction of the available bandwidth. We want to maximize the total bandwidth utilized; i.e., $\sum_{p \in P} x_p$, subject to the constraint that, for each edge e, the sum of the x_p over paths that use e is at most 1.

   - Formulate this problem as a linear program with potentially exponentially many variables.
   - Write down the dual of this linear program and describe how to interpret the variables as edge weights subject to a constraint on the minimum length path from s to t.
   - Argue that the dual linear program can be solved in polynomial time via the ellipsoid algorithm, by describing how to implement a *separation oracle* that, given a purported solution, either verifies that all constraints are satisfied, or returns a violated constraint.