Outline

• Linear programming
  – simplex algorithm
  – LP duality
  – ellipsoid algorithm

* slides from Kevin Wayne

Linear programming

Standard Form LP

"Standard form" LP.
  • Input: real numbers $a_{ij}, c_j, b_i$.
  • Output: real numbers $x_j$.
  • $n =$ # decision variables, $m =$ # constraints.
  • Maximize linear objective function subject to linear inequalities.

$\begin{align*}
(P) & \quad \max \sum_{j=1}^{n} c_j x_j \\
& \text{s.t. } \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \leq i \leq m \\
& \quad x_j \geq 0 \quad 1 \leq j \leq n
\end{align*}$

Brewery Problem: Converting to Standard Form

Original input.

$max \ 13A + 23B \\
\text{s.t. } 5A + 15B \leq 480 \\
4A + 4B \leq 160 \\
35A + 20B \leq 1190 \\
A, B \geq 0$

Standard form.
  • Add slack variable for each inequality.
  • Now a 5-dimensional problem.

$\begin{align*}
(P) & \quad \max c^T x \\
& \text{s.t. } Ax = b \\
& \quad x \geq 0
\end{align*}$

Equivalent Forms

Easy to convert variants to standard form.

$\begin{align*}
(P) & \quad \max c^T x \\
& \text{s.t. } Ax = b \\
& \quad x \geq 0
\end{align*}$

Less than to equality:
  $x + 2y - 3z \leq 17 \quad \Rightarrow \quad x + 2y - 3z = 17, x \geq 0$

Greater than to equality:
  $x + 2y - 3z \geq 17 \quad \Rightarrow \quad x + 2y - 3z = 17, x \geq 0$

Min to max:
  $\min x + 2y - 3z \quad \Rightarrow \quad \max -x - 2y + 3z$

Unrestricted to nonnegative:
  $x \text{ unrestricted} \quad \Rightarrow \quad x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$
Linear programming
geometric perspective

Brewery Problem: Feasible Region

Brewery Problem: Objective Function

Brewery Problem: Geometry

Convexity

Geometric perspective

Convex set. If two points $x$ and $y$ are in the set, then so is $\lambda x + (1-\lambda) y$ for $0 \leq \lambda \leq 1$.

vertex. A point $x$ in the set that can't be written as a strict convex combination of two distinct points in the set.

Observation. LP feasible region is a convex set.

Theorem. If there exists an optimal solution to $(P)$, then there exists one that is a vertex.

Intuition. If $x$ is not a vertex, move in a non-decreasing direction until you reach a boundary. Repeat.
**Suppose**

**Geometric perspective**

**Theorem.** If there exists an optimal solution to (P), then there exists one that is a vertex.

**PF.**

- Suppose \( x \) is an optimal solution that is not a vertex.
- There exist direction \( d \) of \( \text{conv} \{\mathbf{0}\} \), \( \mathbf{0} \) such that \( x + d \not\in P \).
- \( Ax = b \) because \( Ax \not\in d \).
- Assume \( c^T d \geq 0 \) (by taking either \( d \) or \( -d \)).
- Consider \( x + \lambda d \), \( \lambda > 0 \).

**Case 1.** \( \exists \lambda \) such that \( x + \lambda d \not\in P \)

- Increase \( \lambda \) to \( \lambda' \) until first new component of \( x + \lambda' d \) hits \( P \).
- \( x + \lambda' d \) is feasible since \( \lambda' > 0 \) and \( x + \lambda' d \not\in P \).
- \( x + \lambda' d \) has one more zero component than \( x \).
- \( c^T (x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x \).

\( L = 0 \) whenever \( \lambda' = 0 \) because \( x \not\in P \).

**Case 2.** \( \lambda' \geq 0 \) for all \( \lambda' \)

- \( x + \lambda d \) is feasible for all \( \lambda \geq 0 \) since \( (x + \lambda d) \cdot h \) and \( x + \lambda d \not\in P \).
- As \( \lambda \to \infty \), \( c^T (x + \lambda d) \to \infty \) because \( c^T d > 0 \).

\( P \not\subset \mathbb{R} \), choose \( \lambda \) so that case 1 applies.

**Intuition**

**A vertex in** \( \mathbb{R}^n \) **is uniquely specified by** \( m \) **linearly independent equations.**

**Linear programming**

**linear algebraic perspective**

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**Basic Feasible Solution**

**Theorem.** Let \( P = \{ x : Ax = b, x \geq 0 \} \). For \( x \in P \), define \( B = \{ j : x_j > 0 \} \). Then \( x \) is a vertex if \( B \) has linearly independent columns.

**Notation.** Let \( B \) - set of column indices. Define \( B_x \) to be the subset of columns of \( A \) indexed by \( B \).

**Ex.**

\[
A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 18 \end{bmatrix}
\]

\[
x = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B = \{1, 3\}, \quad B_x = \begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix}
\]

**Basic Feasible Solution**

**Theorem.** Let \( P = \{ x : Ax = b, x \geq 0 \} \). For \( x \in P \), define \( B = \{ j : x_j > 0 \} \). Then \( x \) is a vertex if \( B_x \) has linearly dependent columns.

**PF.**

- Assume \( x \) is not a vertex.
- There exist direction \( d \) not equal to \( 0 \) such that \( x + d \not\in P \).
- \( Ad = 0 \) because \( Ax = b \).
- Define \( B' = \{ j : d_j \not\in \text{conv} \{ B \} \} \).
- \( B_x \) has linearly dependent columns since \( d \not\in B \).
- Moreover, \( d_x = 0 \) whenever \( x_j = 0 \) because \( x \not\in P \).
- Thus \( B' \not\subset R \), so \( B_x \) is a submatrix of \( B_x \).
- Therefore, \( B_x \) has linearly dependent columns.
**Basic Feasible Solution**

**Theorem.** Let $P = \{ x : Ax = b, x \geq 0 \}$. For $x \in P$, define $B = \{ j : x_j > 0 \}$. Then $x$ is a vertex if $B$ has linearly independent columns.

**Proof.**
- Assume $A$ has linearly dependent columns.
- There exist $d$ not equal to 0 such that $A \cdot d = 0$.
- Extend $d$ to $\mathbb{R}^n$ by adding 0 components.
- Now, $A \cdot d = 0$ and $d_j = 0$ whenever $x_j = 0$.
- For sufficiently small $\lambda$, $x + \lambda d \in P \Rightarrow x$ is not a vertex.

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**Basic Feasible Solution: Example**

<table>
<thead>
<tr>
<th>Basis</th>
<th>$A \cdot x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${ S_C, S_H, S_M }$</td>
<td>0.32</td>
</tr>
<tr>
<td>${ S_C, S_H }$</td>
<td>1.20</td>
</tr>
<tr>
<td>${ S_C, S_M }$</td>
<td>1.50</td>
</tr>
<tr>
<td>${ S_H, S_M }$</td>
<td>2.00</td>
</tr>
<tr>
<td>${ S_C }$</td>
<td>12.00</td>
</tr>
<tr>
<td>${ S_H }$</td>
<td>25.00</td>
</tr>
<tr>
<td>${ S_M }$</td>
<td>15.00</td>
</tr>
</tbody>
</table>

**Simplex Algorithm**

**Theorem.** Given $P = \{ x : Ax = b, x \geq 0 \}$, $x$ is a vertex if there exists $B \subseteq \{ 1, \ldots, n \}$ such that $|B| = m$ and:
- $A_B$ is nonsingular.
- $x_B = A_B^{-1} b \geq 0$.
- $x_N = 0$.

**Proof.** Augment $A_B$ with linearly independent columns (if needed).

- Assumption. $A_B$ has full row rank.

**Simplex Algorithm: Intuition**

**Simplex Algorithm, [George Dantzig 1947]** Move from BFS to adjacent BFS, without decreasing objective function.

- replace one basic variable with another
- Greedy property. BFS optimal if no adjacent BFS is better.
- Challenge. Number of BFS can be exponential!

**Simplex Algorithm: Initialization**

<table>
<thead>
<tr>
<th>max $Z = 13A + 23B$ s.t. $5A + 15B \leq 480$ $4A + 4B \leq 160$ $35A + 26B \leq 1190$ $A, B, S_C, S_H, S_M \geq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis = ${ S_C, S_H, S_M }$ $d = 0$ $Z = 480$ $S_C = 160$ $S_H = 1190$</td>
</tr>
</tbody>
</table>
In particular:

Thus, optimal objective value

Substitute:

Simplex algorithm:

Q. Why pivot on column 2 (or 1)?
A. Each unit increase in \( b \) increases objective value by \$23.

Q. Why pivot on row 2?
A. Preserves feasibility by ensuring RHS \( \geq 0 \).

Q. When to stop pivoting?
A. When all coefficients in top row are nonpositive.

Q. Why is resulting solution optimal?
A. Any feasible solution satisfies system of equations in tableaux.
   • In particular: \( Z = 800 - S_b - 2 S_a \), \( S_a \geq 0 \), \( S_b \geq 0 \).
   • Thus, optimal objective value \( Z^* = 800 \).
   • Current BFS has value \( 800 \Rightarrow \) optimal.

Simplex Algorithm: Corner Cases

Q. What if min ratio test fails?
Q. How to find initial basis?
Q. How to guarantee termination?

Simplex Algorithm: Optimality

Simplex Algorithm: Pivot 1

Simplex Algorithm: Pivot 1

Simplex Algorithm: Pivot 2

Simplex Algorithm: Optimal

Simplex Algorithm: Pivot 1

Simplex Algorithm: Pivot 1

Simplex Algorithm: Pivot 1

Initial simplex tableaux:

Simplex tableaux corresponding to basis \( \beta \).

Simplex tableaux: Matrix Form
Unboundedness

Q. What happens if min ratio test fails?

A. Unbounded objective function.

\[
\begin{align*}
\text{max } Z & \text{ subject to} \\
2x_1 & = 20x_3 - Z = 2 \\
x_1 & - 4x_2 - 8x_3 \leq 3 \\
x_2 & + 5x_4 - 12x_5 \leq 4 \\
x_3 & + 5x_4 - 6x_5 \leq 5 \\
x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{align*}
\]

Phase I Simplex

Q. How to find initial basis?

A. Solve (P'), starting from basis consisting of all the \( z_i \) variables.

- Case 1: \( \min > 0 \) \((P)\) is infeasible.
- Case 2: \( \min = 0 \), basis has no \( z_i \) variables \( \Rightarrow \) OK to start Phase II.
- Case 3a: \( \min = 0 \), basis has \( z_i \) variables. Pivot \( z_i \) variables out of basis. If successful, start Phase II; else remove linear dependent rows.

Simplex Algorithm: Degeneracy

Degeneracy. New basis, same vertex.

Degenerate pivot. Min ratio = 0.

Simplex Algorithm: Degeneracy

Degeneracy. New basis, same vertex.

Cycling. Infinite loop by cycling through different bases that all correspond to same vertex.

Anti-cycling rules.
- Bland's rule: choose eligible variable with smallest index.
- Random rule: choose eligible variable uniformly at random.
- Lexicographic rule: perturb constraints so nondegenerate.

Lexicographic Rule

Intuition. No degeneracy \( \Rightarrow \) no cycling.

Perturbed problem.

\[
\begin{align*}
\text{(P') max } c^T x & \text{ subject to} \\
& \text{s. t. } d\mathbf{x} = b + \epsilon \\
& \mathbf{x} \geq 0
\end{align*}
\]

Lexicographic rule. Apply perturbation virtually by manipulating \( \epsilon \), symbolically:

\[
17 + 5\epsilon = 17 + \epsilon, 17 + 5\epsilon = 17 + \epsilon
\]

Claim. In perturbed problem, \( x_\epsilon = d_\epsilon (\mathbf{b} + \epsilon) \) is always nonzero.

Pf. The \( j \)-th component of \( x_\epsilon \) is a (nonzero) linear combination of the components of \( \mathbf{b} + \epsilon \) \( \Rightarrow \) contains at least one of the \( \epsilon_i \) terms.

Corollary. No cycling.
Simplex Algorithm: Practice

**Remarkable property.** In practice, simplex algorithm typically terminates after at most $2(m + n)$ pivots.

**Issues.**
- Choose the pivot.
- Maintain sparsity.
- Ensure numerical stability.
- Preprocess to eliminate variables and constraints.

Commercial solvers can solve LPs with millions of variables and tens of thousands of constraints.

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**LP Duality**

**Primal problem.**

\[
(P) \quad \text{max} \quad 13A + 23B \\
\text{s.t.} \quad 5A + 15B \leq 480 \\
4A + 4B \leq 160 \\
35A + 20B \leq 1190 \\
A, B \geq 0
\]

**Goal.** Find a **lower bound** on optimal value.

**Easy.** Any feasible solution provides one.

**Ex 1.** \((A, B) = (34, 0)\) \(\Rightarrow z^* \geq 442\)

**Ex 2.** \((A, B) = (0, 32)\) \(\Rightarrow z^* \geq 736\)

**Ex 3.** \((A, B) = (7.5, 29.5)\) \(\Rightarrow z^* \geq 776\)

**Ex 4.** \((A, B) = (12, 28)\) \(\Rightarrow z^* \geq 800\)

**Recall lower bound.** \((A, B) = (12, 28)\) \(\Rightarrow z^* \geq 800\)

**Combine upper and lower bounds:** \(z^* = 800\).

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**Goal.** Find an **upper bound** on optimal value.

**Ex 1.** Multiply 2\(^{nd}\) inequality by 6: \(24A + 24B \leq 960\).

\(\Rightarrow z^* = 13A + 23B \leq 24A + 24B \leq 960\).

**Goal.** Find an **upper bound** on optimal value.

**Ex 2.** Add 2 times 1\(^{st}\) inequality to 2\(^{nd}\) inequality:

\(\Rightarrow z^* = 13A + 23B \leq 14A + 34B \leq 1120\).
LP Duality

**Primal problem.**

\[
\begin{align*}
\text{(P)} \quad & \text{max} \quad 15A + 23B \\
\text{s.t.} \quad & 5A + 15B \leq 480 \\
& 4A + 4B \leq 160 \\
& 35A + 20B \leq 1190 \\
& A, B \geq 0
\end{align*}
\]

**Idea.** Add nonnegative combination \((C, H, M)\) of the constraints \(s.t.

\[
13A + 23B \leq (5C + 4H + 35M)A + (15C + 4H + 20M)B \\
\leq 480C + 1190M
\]

**Dual problem.** Find best such upper bound.

\[
\begin{align*}
\text{(D)} \quad & \text{min} \quad 480C + 160H + 1190M \\
\text{s.t.} \quad & 5C + 4H + 35M \geq 13 \\
& 15C + 4H + 20M \geq 23 \\
& C, H, M \geq 0
\end{align*}
\]

LP Duality: Economic Interpretation

**Brewer:** find optimal mix of beer and ale to maximize profits.

\[
\begin{align*}
\text{(P)} \quad & \text{max} \quad 13A + 23B \\
\text{s.t.} \quad & 5A + 15B \leq 480 \\
& 4A + 4B \leq 160 \\
& 35A + 20B \leq 1190 \\
& A, B \geq 0
\end{align*}
\]

**Entrepreneur:** buy individual resources from brewer at min cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if \(5C + 4H + 35M < 13\).

**Double Dual**

**Canonical form.**

\[
\begin{align*}
\text{(D')} \quad & \text{max} \quad y^T b \\
\text{s.t.} \quad & A^T y \leq c \\
& y \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{(DD)} \quad & \text{min} \quad c^T z \\
\text{s.t.} \quad & (A^T)^T z \geq b \\
& z \geq 0
\end{align*}
\]

**Taking Duals**

LP dual recipe.

**Pf.** Rewrite LP in standard form and take dual.

**Strong duality**

May 20, 2014 CS38 Lecture 15
LP Strong Duality

Theorem. [Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947]
For $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, if (P) and (D) are nonempty, then $\max = \min$.

(P) $\max \mathbf{c}^T \mathbf{x}$
\[ \text{s. t. } \mathbf{A}^T \mathbf{x} \leq \mathbf{b} \]
\[ \mathbf{x} \geq 0 \]

(D) $\min \mathbf{y}^T \mathbf{b}$
\[ \text{s. t. } \mathbf{A} \mathbf{y} \geq \mathbf{c} \]
\[ \mathbf{y} \geq 0 \]

Generalizes:
- Dilworth's theorem.
- König-Egerváry theorem.
- Max-flow min-cut theorem.
- von Neumann's minimax theorem.
- ...

Proof. [ahead]