1. [worth 6 pts] This problem concerns that language TILE, defined as follows. Informally, an instance is a collection of $k$ tile types, together with a list of horizontally compatible pairs of tile types, and a list of vertically compatible pairs of tile types. An $n \times n$ tiling is a placement of tiles into an $n \times n$ grid, so that every pair of horizontally adjacent tiles is horizontally compatible, and every pair of vertically adjacent tiles is vertically compatible; in addition we require that the tile in the upper left corner is tile type 1. The language TILE consists of all those instances for which there exists an $n \times n$ tiling for all $n \geq 0$.

Formally, the language TILE is the set of those tuples

$$\langle k, H \subseteq [k] \times [k], V \subseteq [k] \times [k]\rangle$$

for which the following holds. For all $n \geq 1$ there exists a function $f : [n] \times [n] \rightarrow [k]$ for which:

- $f(1, 1) = 1$, and
- $(f(x, y), f(x, y + 1)) \in H$ for all $1 \leq x \leq n$, and $1 \leq y \leq n - 1$, and
- $(f(x, y), f(x + 1, y)) \in V$ for all $1 \leq x \leq n - 1$ and $1 \leq y \leq n$.

Here, $[n]$ is shorthand for the set $\{1, 2, 3, \ldots, n\}$. Prove that TILE is undecidable by giving a reduction from $\overline{\text{HALT}}$ (the complement of the language HALT). In other words, give a function $R$ mapping instances of $\overline{\text{HALT}}$ to instances of TILE, with the property that $R(\langle M, w \rangle)$ is in the language TILE if and only if $\langle M, w \rangle$ is in the language $\overline{\text{HALT}}$. Hint: it will be helpful to “name” some of your tiles with triplets of symbols.

2. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \rightarrow V_2$ such that $(u, v) \in E_1 \iff (f(u), f(v)) \in E_2$. For a given graph $H$, define the following language:

$$\text{CONTAINS}_H = \{G : G \text{ contains a subgraph isomorphic to } H\}.$$ 

Here by “subgraph” we mean a subset of $G$’s vertices together with all of $G$’s edges on that subset of vertices – often called an “induced subgraph.” Prove that for every $H$, $\text{CONTAINS}_H$ is in P.
3. Show that the following problem is in P:

\[
\text{UNARY SUBSET SUM} = \left\{ (1^B, x_1, x_2, \ldots, x_n) : \exists \text{ a multiset } I \text{ of } [n] \text{ for which } \sum_{i \in I} x_i = B \right\}.
\]

Here the \(x_i\) are all positive integers, as is \(B\), and \([n]\) is shorthand for the set \(\{1, 2, 3, \ldots, n\}\). The notation \(1^B\) means a string of \(B\) ones, which is the representation of \(B\) in unary. Hint: solve the problem for all \(B' \leq B\).