1. [worth 6 pts] This problem concerns that language TILE, defined as follows. Informally, an instance is a collection of \( k \) tile types, together with a list of horizontally compatible pairs of tile types, and a list of vertically compatible pairs of tile types. An \( n \times n \) tiling is a placement of tiles into an \( n \times n \) grid, so that every pair of horizontally adjacent tiles is horizontally compatible, and every pair of vertically adjacent tiles is vertically compatible; in addition we require that the tile in the upper left corner is tile type 1. The language TILE consists of all those instances for which there exists an \( n \times n \) tiling for all \( n \geq 0 \).

Formally, the language TILE is the set of those tuples

\[
\langle k, H \subseteq [k] \times [k], V \subseteq [k] \times [k] \rangle
\]

for which the following holds. For all \( n \geq 1 \) there exists a function \( f : [n] \times [n] \to [k] \) for which:

- \( f(1, 1) = 1 \), and
- \( (f(x, y), f(x, y + 1)) \in H \) for all \( 1 \leq x \leq n \), and \( 1 \leq y \leq n - 1 \), and
- \( (f(x, y), f(x + 1, y)) \in V \) for all \( 1 \leq x \leq n - 1 \) and \( 1 \leq y \leq n \).

Here, \([n]\) is shorthand for the set \( \{1, 2, 3, \ldots, n\} \). Prove that TILE is undecidable by giving a reduction from \( \overline{\text{HALT}} \) (the complement of the language HALT). In other words, give a function \( R \) mapping instances of \( \overline{\text{HALT}} \) to instances of TILE, with the property that \( R(\langle M, w \rangle) \) is in the language TILE if and only if \( \langle M, w \rangle \) is in the language \( \overline{\text{HALT}} \). Hint: it will be helpful to “name” some of your tiles with triplets of symbols.

2. Two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are isomorphic if there is a bijection \( f : V_1 \to V_2 \) such that \( (u, v) \in E_1 \iff (f(u), f(v)) \in E_2 \). For a given graph \( H \), define the following language:

\[
\text{CONTAINS}_H = \{ G : G \text{ contains a subgraph isomorphic to } H \}.
\]

Here by “subgraph” we mean a subset of \( G \)’s vertices together with all of \( G \)’s edges on that subset of vertices – often called an “induced subgraph.” Prove that for every \( H \), \( \text{CONTAINS}_H \) is in \( P \).
3. Show that the following problem is in P:

\[
\text{UNARY SUBSET SUM} = \left\{(1^B, x_1, x_2, \ldots, x_n) : \exists \text{ a multiset } I \text{ of } [n] \text{ for which } \sum_{i \in I} x_i = B \right\}.
\]

Here the \(x_i\) are all positive integers, as is \(B\), and \([n]\) is shorthand for the set \(\{1, 2, 3, \ldots, n\}\). The notation \(1^B\) means a string of \(B\) ones, which is the representation of \(B\) in unary. Hint: solve the problem for all \(B' \leq B\).