

## Midterm

Out: January 31

Due: February 7

**This is a midterm.** You may consult only the course notes and the text (Sipser). *You may not collaborate.* The full honor code guidelines can be found in the course syllabus.

There are 5 problems on 2 pages. Please attempt all problems. **To facilitate grading, please turn in each problem on a separate sheet of paper and put your name on each sheet. Do not staple the separate sheets.** Good luck!

- Identify each of the following languages as either (i) regular, (ii) context-free but not regular, or (iii) not context-free. For each language, prove that your classification is correct, using the techniques we have developed in this course.

(a)  $L_1 = \{a^i b^j c^i d^j : i, j \geq 0\}$ .

(b)  $L_2 = \{a^i b^j c^j d^i : i, j \geq 0\}$ .

(c)  $L_3 = \{a^i b^j c^k : i = j = k \text{ or } i > 1000\}$ .

- Identify each of the following languages as either decidable or undecidable, and prove that your classification is correct, using the techniques we have developed in this course. Recall that for a context free grammar  $G$ , we denote by  $L(G)$  the language it describes, and similarly for a regular expression  $E$ , we denote by  $L(E)$  the language it describes.

(a)  $\text{CFL-IN-REG} = \{(G, E) : G \text{ is a CFG, } E \text{ is a regular expression, and } L(G) \subseteq L(E)\}$

(b)  $\text{REG-IN-CFL} = \{(E, G) : G \text{ is a CFG, } E \text{ is a regular expression, and } L(E) \subseteq L(G)\}$

Hint: you may wish to use the fact that the intersection of a context free language and a regular language is context-free (Sipser problem 2.18a).

- Two (disjoint) languages  $L_1$  and  $L_2$  are called *recursively separable* if there is a decidable language  $D$  for which  $L_1 \cap D = \emptyset$  and  $L_2 \subseteq D$ ; they are *recursively inseparable* if no such decidable language  $D$  exists. Convince yourself that an undecidable language and its complement are recursively inseparable.

Consider the following languages:

$$L_1 = \{\langle M \rangle : M \text{ halts and accepts input } \langle M \rangle\}$$

$$L_2 = \{\langle M \rangle : M \text{ halts and rejects input } \langle M \rangle\}$$

Prove that  $L_1$  and  $L_2$  are recursively inseparable. Hint: your proof will probably involve supplying a Turing Machine its own description as input.

4. A *right-linear* CFG is a context-free grammar in which every production has the form

- $A \rightarrow xB$ , or
- $A \rightarrow x$ ,

where  $A$  and  $B$  are non-terminals, and  $x$  can be any string of terminals. A CFG is *linear* if productions of the form  $A \rightarrow Bx$  are allowed in addition to the two types of productions in a right-linear CFG.

- (a) Prove that every language generated by a right-linear CFG is regular.
- (b) Prove that every regular language is generated by some right-linear CFG.
- (c) Give a *linear* CFG that generates the following non-regular language over the alphabet  $\Sigma = \{0, 1\}$ :

$$L = \{w : w \text{ is a palindrome}\},$$

and prove that your grammar indeed generates exactly  $L$  (i.e., prove that every string in  $L$  is generated by your grammar, and prove that every string generated by your grammar is in  $L$ ).

5. Given two languages  $A$  and  $B$  over an alphabet  $\Sigma$ , define a third language, denoted  $A/B$ , as follows:

$$A/B = \{x : \text{there exists some } y \in B \text{ for which } xy \in A\}.$$

Prove that if  $A$  and  $B$  are both RE, then  $A/B$  is RE.