CS21 Decidability and Tractability

Lecture 9
January 27, 2025

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Summary

- Nondeterministic Pushdown Automata (NPDA)
- Context-Free Grammars (CFGs) describe Context-Free Languages (CFLs)
  - terminals, non-terminals
  - productions
  - yields, derivations
  - parse trees

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Summary

  - grouping determined by grammar
  - Chomsky Normal Form (CNF)

- NDPAs and CFGs are equivalent

- CFL Pumping Lemma is used to show certain languages are not CFLs

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Summary

- deterministic PDAs recognize DCFLs
- DCFLs are closed under complement

- there is an efficient algorithm (based on dynamic programming) to determine if a string x is generated by a given grammar G

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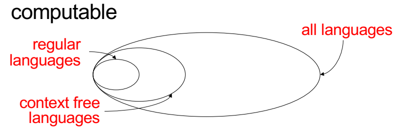
So far…

- several models of computation
  - finite automata
  - pushdown automata
- fail to capture our intuitive notion of what is computable



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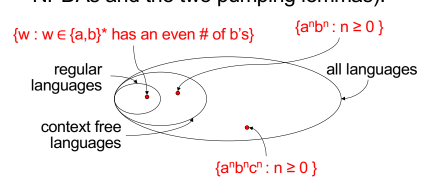
So far…

- We proved (using constructions of FA and NPDAs and the two pumping lemmas):



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A more powerful machine

- limitation of NPDA related to fact that their memory is stack-based (last in, first out)
- What is the simplest alteration that adds general-purpose “memory” to our machine?
- Should be able to recognize, e.g., $\{a^n b^n c^n : n \geq 0\}$

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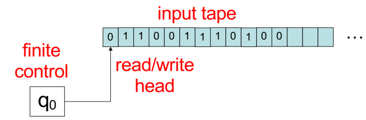
Turing Machines



- New capabilities:
  - infinite tape
  - can read OR write to tape
  - read/write head can move left and right

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Turing Machine

- Informal description:
  - input written on left-most squares of tape
  - rest of squares are blank
  - at each point, take a step determined by
    - current symbol being read
    - current state of finite control
  - a step consists of
    - writing new symbol
    - moving read/write head left or right
    - changing state

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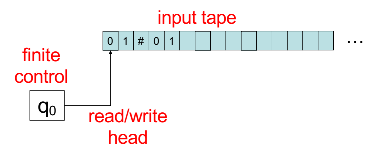
Example Turing Machine

language L = {w#w : w ∈ {0,1}*}



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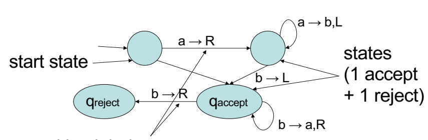
Turing Machine diagrams



transition label: (tape symbol read → tape symbol written, direction moved)

“_” means blank tape square

- a → R means “read a, move right”
- a → L means “read a, move left”
- a → b, R means “read a, write b, move right

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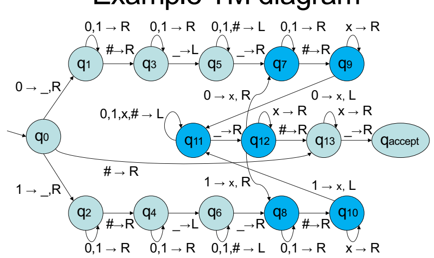
Example TM diagram



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TM formal definition

- A TM is a 7-tuple
  $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where:
  - Q is a finite set called the states
  - Σ is a finite set called the input alphabet
  - Γ is a finite set called the tape alphabet
  - $\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is a function called the transition function
  - q_0 is an element of Q called the start state
  - q_{accept}, q_{reject} are the accept and reject states

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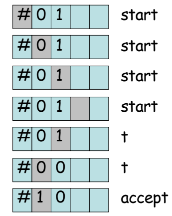
Example TM operation



program for "binary successor"

q	σ	δ(q,σ)
start	0	(start, 0, R)
start	1	(start, 1, R)
start	_	(t, _, L)
start	#	(start, #, R)
t	0	(accept, 1, -)
t	1	(t, 0, L)
t	#	(accept, #, R)

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TM configurations

- At every step in a computation, a TM is in a configuration determined by:
  - the contents of the tape
  - the state
  - the location of the read/write head
- next step completely determined by current configuration
- shorthand: string uqv with $u,v \in \Gamma^*$, $q \in Q$

meaning:
• tape contents: uv followed by blanks
• in state q
• reading first symbol of v

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TM configurations

- configuration C_1 yields configuration C_2 if TM can legally* move from C_1 to C_2 in 1 step
  - notation: $C_1 \Rightarrow C_2$
  - also: "yields in 1 step" notation: $C_1 \Rightarrow^1 C_2$
  - "yields in k steps" notation: $C_1 \Rightarrow^k C_2$
  if there exists configurations $D_1, D_2, \ldots D_{k-1}$ for which $C_1 \Rightarrow D_1 \Rightarrow D_2 \Rightarrow \ldots \Rightarrow D_{k-1} \Rightarrow C_2$
  - also: "yields in some # of steps" $(C_1 \Rightarrow^* C_2)$

*Convention: TM halts upon entering q_{accept}, q_{reject}

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TM configurations

- Formal definition of "yields":
  $ua q_i bv \Rightarrow u q_j acv$
  if $\delta(q_i, b) = (q_j, c, L)$, and
  $ua q_i bv \Rightarrow uac q_j v$
  if $\delta(q_i, b) = (q_j, c, R)$
- two special cases:
  - left end: $q_i bv \Rightarrow q_j cv$ if $\delta(q_i, b) = (q_j, c, L)$
  - right end: $ua q_i$ same as $ua q_i \_$

$u,v \in \Gamma^*$
$a,b,c \in \Gamma$
$q_i, q_j \in Q$

$(q_i \neq q_{accept}, q_{reject})$

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TM acceptance

- start configuration: $q_0 w$ (w is input)
- accepting config.: any config.with state q_{accept}
- rejecting config.: any config. with state q_{reject}

TM M accepts input w if there exist configurations $C_1, C_2, \ldots, C_k$
  - C_1 is start configuration of M on input w
  - $C_i \Rightarrow C_{i+1}$ for i = 1, 2, 3, …, k-1
  - C_k is an accepting configuration

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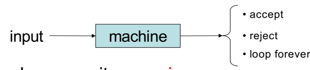
Deciding and Recognizing



- TM M:
  - L(M) is the language it recognizes
  - if M rejects every x ∉ L(M) it decides L
  - set of languages recognized by some TM is called Turing-recognizable or recursively enumerable (RE)
  - set of languages decided by some TM is called Turing-decidable or decidable or recursive

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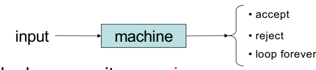
Deciding and Recognizing



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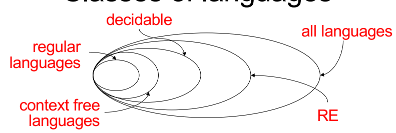
Classes of languages



- We know: regular ⊆ CFL (proper containment)
- CFL ⊆ decidable
  - proof?
  - decidable ⊆ RE ⊆ all languages
  - proof?

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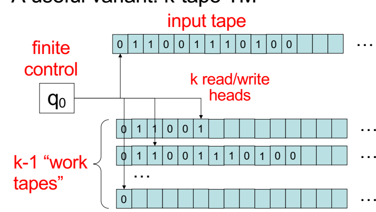
Multitape TMs

- A useful variant: k-tape TM



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Multitape TMs

- Informal description of k-tape TM:
  - input written on left-most squares of tape #1
  - rest of squares are blank on all tapes
  - at each point, take a step determined by
    - current k symbols being read on k tapes
    - current state of finite control
  - a step consists of
    - writing k new symbols on k tapes
    - moving each of k read/write heads left or right
    - changing state

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Multitape TM formal definition

- A TM is a 7-tuple
  $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where:
  - everything is the same as a TM except the transition function:

$$\delta: Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$$

$\delta(q_i, a_1, a_2, \ldots, a_k) = (q_j, b_1, b_2, \ldots, b_k, L, R, \ldots, L) =$
"in state q_i, reading $a_1, a_2, \ldots, a_k$ on k tapes, move to state q_j, write $b_1, b_2, \ldots, b_k$ on k tapes, move L, R on k tapes as specified."

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Multitape TMs

Theorem: every k-tape TM has an equivalent single-tape TM.

Proof:

– Idea: simulate k-tape TM on a 1-tape TM.