NPDA, CFG equivalence

Theorem: a language $L$ is recognized by a
NPDA iff $L$ is described by a CFG.

Must prove two directions:

$(\Rightarrow)$ $L$ is recognized by a NPDA implies $L$ is
described by a CFG.

$(\Leftarrow)$ $L$ is described by a CFG implies $L$ is
recognized by a NPDA.

Proof of $(\Rightarrow)$: $L$ is described by a CFG
implies $L$ is recognized by a NPDA.

1. we’d like to non-deterministically guess the
derivation, forming it on the stack
2. then scan the input, popping matching
symbol off the stack at each step
3. accept if we get to the bottom of the stack at
the end of the input.

What is wrong with this approach?

• informal description of construction:
  - place $\$ and start symbol $S$ on the stack
  - repeat:
    • if the top of the stack is a non-terminal $A$, pick a
      production with $A$ on the rhs and substitute the rhs
      for $A$ on the stack
    • if the top of the stack is a terminal $b$, read $b$ from
      the tape, and pop $b$ from the stack.
    • if the top of the stack is $\$, enter the accept state.
NPDA, CFG equivalence

Proof of (⇒): L is recognized by a NPDA implies L is described by a CFG.

- harder direction
- first step: convert NPDA into "normal form":
  - single accept state
  - empties stack before accepting
  - each transition either pushes or pops a symbol

- main idea: non-terminal $A_{p,q}$ generates exactly the strings that take the NPDA from state $p$ (w/ empty stack) to state $q$ (w/ empty stack)

- then $A_{\text{start, accept}}$ generates all of the strings in the language recognized by the NPDA.

NPDA P = $(Q, \Sigma, \Gamma, \delta, \text{start}, \{\text{accept}\})$

CFG G:
- non-terminals $V = \{A_{p,q} : p, q \in Q\}$
- start variable $A_{\text{start, accept}}$
- productions:
  - for every $p, r, q \in Q$, add the rule $A_{p,q} \rightarrow A_{p,r}A_{r,q}$
NPDA, CFG equivalence

1. if $A_{p,q}$ generates string $x$, then $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack)
   - induction on length of derivation of $x$.
   - base case: 1 step derivation, must have only terminals on rhs. In $G$, must be production of form $A_{p,p} \rightarrow \varepsilon$.

2. if $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack), then $A_{p,q}$ generates string $x$
   - induction on # of steps in P’s computation
   - base case: 0 steps. starts and ends at same state $p$. only has time to read empty string $\varepsilon$.
   - $G$ contains $A_{p,p} \rightarrow \varepsilon$. 

two claims to verify correctness:

1. if $A_{p,q}$ generates string $x$, then $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack)
2. if $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack), then $A_{p,q}$ generates string $x$
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2. if x can take NPDA P from state p (w/ empty stack) to q (w/ empty stack), then $A_{p,q}$ generates string x

- induction step: assume true for computations of length at most k, prove for length k+1.
- if stack becomes empty sometime in the middle of the computation (at state r)
  - y is read going from state p to r ($A_{p,r} \rightarrow y$)
  - z is read going from state r to q ($A_{r,q} \rightarrow z$)
  - conclude: $A_{p,r} \rightarrow A_{p,q} \rightarrow yz = x$

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2. if x can take NPDA P from state p (w/ empty stack) to q (w/ empty stack), then $A_{p,q}$ generates string x

- if stack becomes empty only at beginning and end of computation.
  - first step: state p to r, read a, push d ($A_{p,r} \rightarrow y$)
  - go from state r to s, read string y ($A_{r,s} \rightarrow y$)
  - last step: state s to q, read b, pop d ($A_{s,q} \rightarrow x$)
  - conclude: $A_{p,q} \rightarrow aA_{r,s}b \rightarrow ayb = x$

Chomsky Normal Form

- Useful to deal only with CFGs in a simple normal form
- Most common: Chomsky Normal Form (CNF)
- Definition: every production has form $A \rightarrow BC$ or $S \rightarrow \varepsilon$ or $A \rightarrow a$ where $A, B, C$ are any non-terminals (and $B, C$ are not $S$) and $a$ is any terminal.

Theorem: Every CFL is generated by a CFG in Chomsky Normal Form.

Proof: exercise or in book…

Deciding CFLs

- Useful to have an efficient algorithm to decide whether string x is in given CFL
  - e.g., programming language often described by CFG, determine if string is valid program.
  - If CFL recognized by deterministic PDA, just simulate the PDA.
  - but not all CFLs are (homework)…
- Can simulate NPDA, but this takes exponential time in the worst case.
Deciding CFLs

- Convert CFG into Chomsky Normal Form
- Parse tree for string $x$ generated by nonterminal $A$:
  
  ![Parse tree diagram]

  If $A \rightarrow x$ (k > 1) then there must be a way to split $x$:
  - $A \rightarrow BC$ is a production and
  - $B \rightarrow y$ and $C \Rightarrow z$ for $i, j < k$

Deciding CFLs

- An algorithm: $IsGenerated(x, A)$
  - if $|x| = 1$, then return YES if $A \rightarrow x$ is a production, else return NO
  - for all $n-1$ ways of splitting $x = yz$
    - for all $m$ productions of form $A \rightarrow BC$
      - if $IsGenerated(y, B)$ and $IsGenerated(z, C)$, return YES
      - return NO

- worst case running time?