CS21 Decidability and Tractability
Lecture 7
January 20, 2023

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma

Outline
• equivalence of NPDAs and CFGs
• non-Context-Free languages via the CFL Pumping Lemma
Some facts about CFLs

- CFLs are closed under
  - union (proof?)
  - concatenation (proof?)
  - star (proof?)

- Every regular language is a CFL
  - proof?

NPDA, CFG equivalence

Theorem: a language $L$ is recognized by a NPDA if and only if $L$ is described by a CFG.

Must prove two directions:

($\Rightarrow$) $L$ is recognized by a NPDA implies $L$ is described by a CFG.

($\Leftarrow$) $L$ is described by a CFG implies $L$ is recognized by a NPDA.

NPDA, CFG equivalence

Proof of ($\Leftarrow$): $L$ is described by a CFG implies $L$ is recognized by a NPDA.

1. we’d like to non-deterministically guess the derivation, forming it on the stack
2. then scan the input, popping matching symbol off the stack at each step
3. accept if we get to the bottom of the stack at the end of the input.

what is wrong with this approach?

NPDA, CFG equivalence

- only have access to top of stack
- combine steps 1 and 2:
  - allow to match stack terminals with tape during the process of producing the derivation on the stack

NPDA, CFG equivalence

- informal description of construction:
  - place $\$ and start symbol $S$ on the stack
  - repeat:
    - if the top of the stack is a non-terminal $A$, pick a production with $A$ on the lhs and substitute the rhs for $A$ on the stack
    - if the top of the stack is a terminal $b$, read $b$ from the tape, and pop $b$ from the stack.
    - if the top of the stack is $\$$, enter the accept state.
NPDA, CFG equivalence

- main idea: non-terminal $A_{p,q}$ generates exactly the strings that take the NPDA from state $p$ (w/ empty stack) to state $q$ (w/ empty stack)

- then $A_{\text{start}, \text{accept}}$ generates all of the strings in the language recognized by the NPDA.

NPDA, CFG equivalence

- Two possibilities to get from state $p$ to $q$:
  - stack height

NPDA, CFG equivalence

- Two possibilities to get from state $p$ to $q$:
  - stack height

NPDA, CFG equivalence

Proof of $(\Rightarrow)$: $L$ is recognized by a NPDA implies $L$ is described by a CFG.

- harder direction
- first step: convert NPDA into “normal form”:
  - single accept state
  - empties stack before accepting
  - each transition either pushes or pops a symbol
NPDA, CFG equivalence

- NPDA $P = (Q, \Sigma, \Gamma, \delta, \text{start}, \{\text{accept}\})$
- CFG $G$:
  - non-terminals $V = \{A_{p,q} : p, q \in Q\}$
  - start variable $A_{\text{start}}$
  - productions:
    - for every $p, r, s, q \in Q$, $d \in \Gamma$ and $a, b \in (\Sigma \cup \{\varepsilon\})$ where $(r, d) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, d)$, add the rule
      $A_{p,q} \rightarrow aA_{r,s}b$

1. if $A_{p,q}$ generates string $x$, then $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack)
2. if $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack), then $A_{p,q}$ generates string $x$

NPDA, CFG equivalence

1. if $A_{p,q}$ generates string $x$, then $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack)
   - induction on length of derivation of $x$.
   - base case: 1 step derivation. must have only terminals on rhs. In $G$, must be production of form $A_{p,p} \rightarrow \varepsilon$.

2. if $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack), then $A_{p,q}$ generates string $x$
   - induction on # of steps in P's computation.
   - base case: 0 steps. starts and ends at same state $p$. only has time to read empty string $\varepsilon$.
   - $G$ contains $A_{p,p} \rightarrow \varepsilon$. 

Note: The diagrams and some text are partially obscured or not entirely visible in the image.
2. if $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack), then $A_{p,q}$ generates string $x$

- induction step. assume true for computations of length at most $k$, prove for length $k+1$.
- if stack becomes empty sometime in the middle of the computation (at state $r$)
  - $y$ is read going from state $p$ to $r$ ($A_{p,r} \rightarrow^* y$)
  - $z$ is read going from state $r$ to $q$ ($A_{r,q} \rightarrow^* z$)
  - conclude: $A_{p,q} \rightarrow A_{p,r} A_{r,q} \rightarrow^* yz = x$

2. if $x$ can take NPDA $P$ from state $p$ (w/ empty stack) to $q$ (w/ empty stack), then $A_{p,q}$ generates string $x$

- if stack becomes empty only at beginning and end of computation.
  - first step: state $p$ to $r$, read $a$, push $d$
  - go from state $r$ to $s$, read string $y$ ($A_{r,s} \rightarrow^* y$)
  - last step: state $s$ to $q$, read $b$, pop $d$
  - conclude: $A_{p,q} \rightarrow aA_{r,s} b \rightarrow^* ayb = x$