# CS21 Decidability and Tractability

Lecture 5
January 13, 2023

1

## Outline

- Non-regular languages: Pumping Lemma
- Pushdown Automata
- Context-Free Grammars and Languages



2

## Non-regular languages

**Pumping Lemma**: Let L be a regular language. There exists an integer p ("pumping length") for which every w ∈ L with |w| ≥ p can be written as

w = xyz such that

1. for every i ≥ 0, $xy^iz \in L$ , and
2. |y| > 0, and
3. |xy| ≤ p.



3

## Non-regular languages

- Using the Pumping Lemma to prove L is not regular:
  - assume L is regular
  - then there exists a pumping length p
  - select a string w ∈ L of length at least p
  - argue that for every way of writing w = xyz that satisfies (2) and (3) of the Lemma, pumping on y yields a string not in L.
  - contradiction.



4

## Pumping Lemma Examples

- Theorem: $L = \{0^i1^j : i > j\}$ is not regular.
- Proof:
  - let p be the pumping length for L
  - choose $w = 0^{p+1}1^p$

    $w = \underbrace{000000000\ldots0}_{p+1}\underbrace{1111111\ldots1}_{p}$

  - w = xyz, with |y| > 0 and |xy| ≤ p.



5

## Pumping Lemma Examples

- 1 possibility:

  $w = \underbrace{000000}_{x}\underbrace{000}_{y}\underbrace{0\ldots0111111111\ldots1}_{z}$

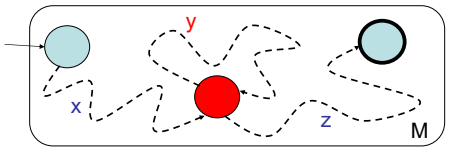
- pumping on y gives strings in the language (?)
- this seems like a problem…
- Lemma states that for every $i \geq 0$, $xy^iz \in L$
- $xy^0z$ not in L. So L not regular.



6

## Proof of the Pumping Lemma

– Let M be a FA that recognizes L.
– Set p = number of states of M.
– Consider w ∈ L with |w| ≥ p. On input w, M must go through *at least* p+1 states. There must be a repeated state (among first p+1).






7

## FA Summary

- A “problem” is a language
- A “computation” receives an input and either accepts, rejects, or loops forever.
- A “computation” recognizes a language (it may also decide the language).
- Finite Automata perform simple computations that read the input from left to right and employ a finite memory.



8

## FA Summary

- The languages recognized by FA are the regular languages.
- The regular languages are closed under union, concatenation, and star.
- Nondeterministic Finite Automata may have several choices at each step.
- NFAs recognize exactly the same languages that FAs do.



9

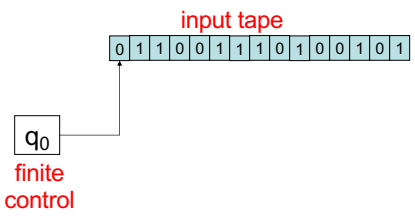
## FA Summary

- Regular expressions are languages built up from the operations union, concatenation, and star.
- Regular expressions describe exactly the same languages that FAs (and NFAs) recognize.
- Some languages are not regular. This can be proved using the Pumping Lemma.



10

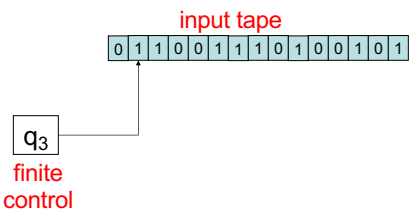
## Machine view of FA






11

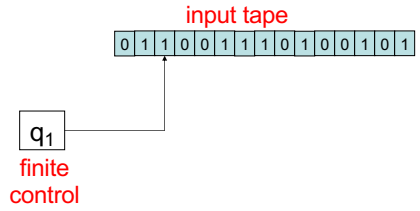
## Machine view of FA






12

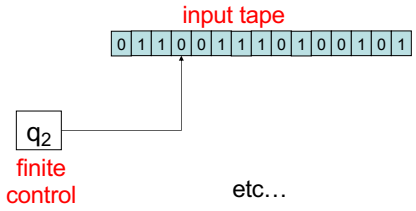
## Machine view of FA






13

## Machine view of FA






14

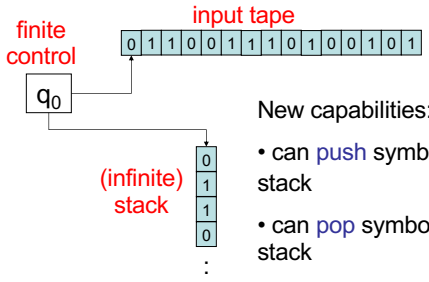
## A more powerful machine

- limitation of FA related to fact that they can only "remember" a bounded amount of information
- What is the simplest alteration that adds unbounded "memory" to our machine?
- Should be able to recognize, e.g., $\{0^n1^n: n \geq 0\}$



15

## Pushdown Automata



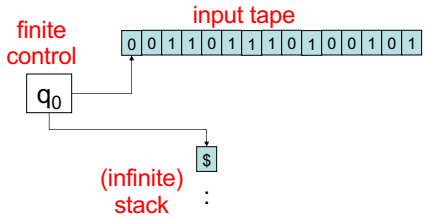

New capabilities:

- can push symbol onto stack
- can pop symbol off of stack



16

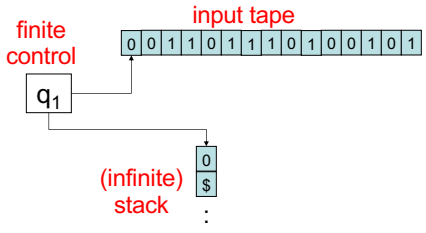
## Pushdown Automata






17

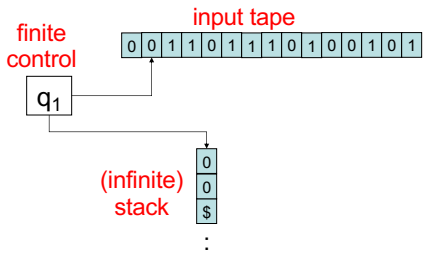
## Pushdown Automata






18

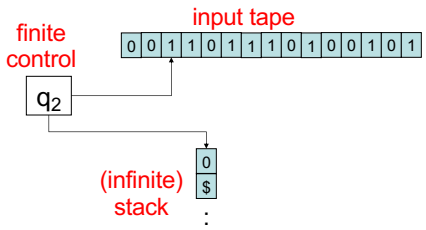
## Pushdown Automata






19

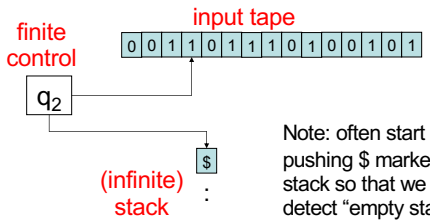
## Pushdown Automata






20

## Pushdown Automata




Note: often start by pushing $ marker onto stack so that we can detect "empty stack"



21

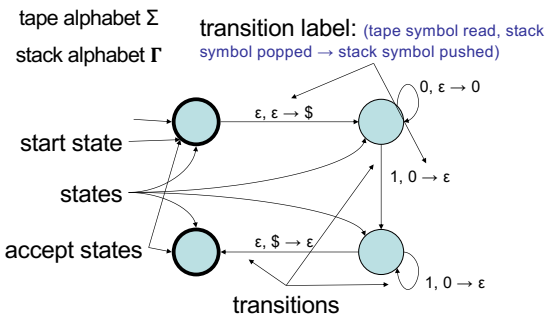
## Pushdown Automata (PDA)

- We will define nondeterministic pushdown automata immediately
  - potentially several choices of "next step"
- Deterministic PDA defined later
  - weaker than NPDA
- Two ways to describe NPDA
  - diagram
  - formal definition



22

## NPDA diagram






23

## NPDA operation

- Taking a transition labeled:

  a, b → c

  – a ∈ (Σ ∪ {ε})
  – b,c ∈ (Γ ∪ {ε})

  – read a from tape, or don't read from tape if a = ε
  – pop b from stack, or don't pop from stack if b = ε
  – push c onto stack, or don't push onto stack if c = ε

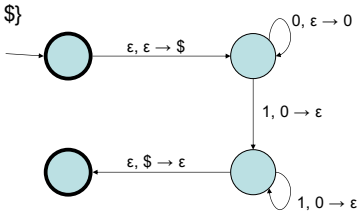


24

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 1 Stack contents: $

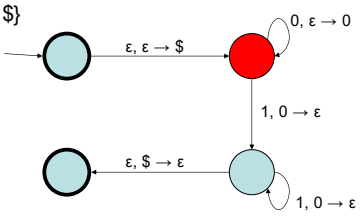


25

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 1 Stack contents: 0 $

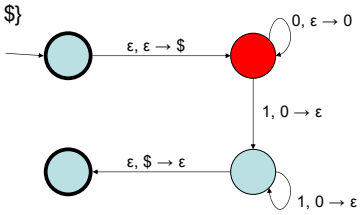


26

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 1 Stack contents: 0 0 $

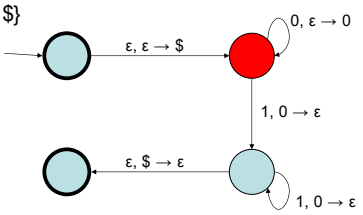


27

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 1 Stack contents: 0 0 $

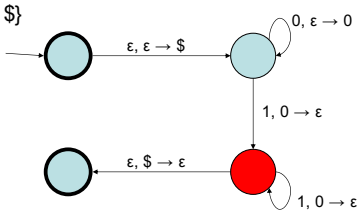


28

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 1 Stack contents: 0 $

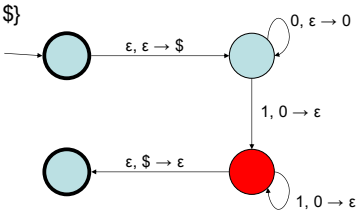


29

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 1 accepted Stack contents: $

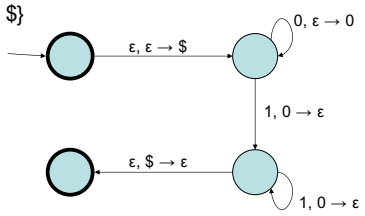


30

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 Stack contents: $

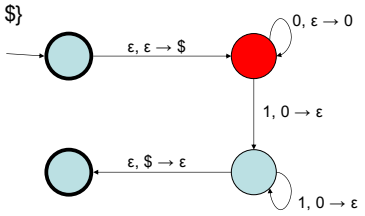


31

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 Stack contents: 0 $

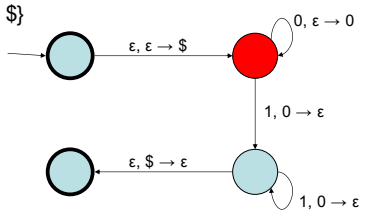


32

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 Stack contents: 0 0 $

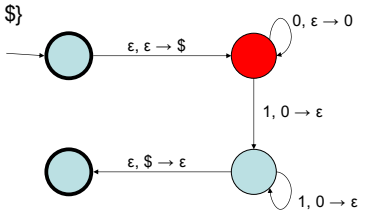


33

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 Stack contents: 0 0 $

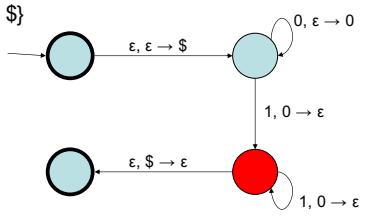


34

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- tape: 0 0 1 Stack contents: 0 $

not accepted

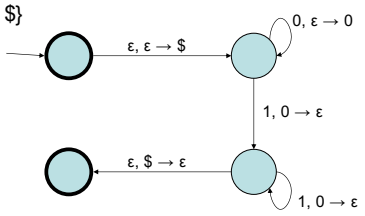


35

## Example NPDA

Σ = {0, 1}

Γ = {0, 1, $}




- What language does this NPDA accept?



36

## Formal definition of NPDA

- A NPDA is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where:
  - Q is a finite set called the states
  - Σ is a finite set called the tape alphabet
  - Γ is a finite set called the stack alphabet
  - $\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q \times (\Gamma \cup \{\varepsilon\}))$ is a function called the transition function
  - $q_0$ is an element of Q called the start state
  - F is a subset of Q called the accept states



37

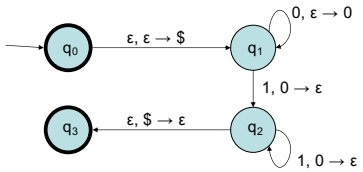
## Formal definition of NPDA

- NPDA M = $(Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts string $w \in \Sigma^*$ if w can be written as
  $w_1 w_2 w_3 \ldots w_m \in (\Sigma \cup \{\varepsilon\})^*$, and
- there exist states $r_0, r_1, r_2, \ldots, r_m$, and
- there exist strings $s_0, s_1, \ldots, s_m$ in $(\Gamma \cup \{\varepsilon\})^*$
  - $r_0 = q_0$ and $s_0 = \varepsilon$
  - $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$, $s_{i+1} = bt$ for some $t \in \Gamma^*$
  - $r_m \in F$



38

## Example of formal definition




- $Q = \{q_0, q_1, q_2, q_3\}$
- $\Sigma = \{0,1\}$
- $\Gamma = \{0, 1, \$\}$
- $F = \{q_0, q_3\}$

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$
- $\delta(q_1, 1, 0) = \{(q_2, \varepsilon)\}$
- $\delta(q_2, 1, 0) = \{(q_2, \varepsilon)\}$
- $\delta(q_2, \varepsilon, \$) = \{(q_3, \varepsilon)\}$

other values of $\delta(\bullet, \bullet, \bullet)$ equal {}



39