CS21
Decidability and Tractability

Lecture 24

March 5, 2018

Outline

- challenges to the extended Church-Turing Thesis
  - randomized computation
  - quantum computation

- today: course review

Course Review

Review

- Highest level: 2 main points

1. Decidability
   - problem solvable by an algorithm = problem is decidable
   - some problems are not decidable (e.g. HALT)

Review

- Highest level: 2 main points

2. Tractability
   – problem solvable in polynomial time = problem is tractable
   – some problems are not tractable (EXP-complete problems)
   – huge number of problems are likely not to be tractable (NP-complete problems)

Review

- Important ideas
  - “problem” formalized as language
    - language = set of strings
  - “computation” formalized as simple machine
    - finite automata
    - pushdown automata
    - Turing Machine
  - “power” of machine formalized as the set of languages it recognizes

Review

- Important ideas (continued):
  - simulation used to show one model at least as powerful as another
  - diagonalization used to show one model strictly more powerful than another
    - also Pumping Lemma
  - reduction used to compare one problem to another

Review

- Important ideas (continued):
  - complexity theory investigates the resources required to solve problems
    - time, space, others…
  - complexity class = set of languages
  - language L is C-hard if every problem in C reduces to L
  - language L is C-complete if L is C-hard and L is in C.

Review

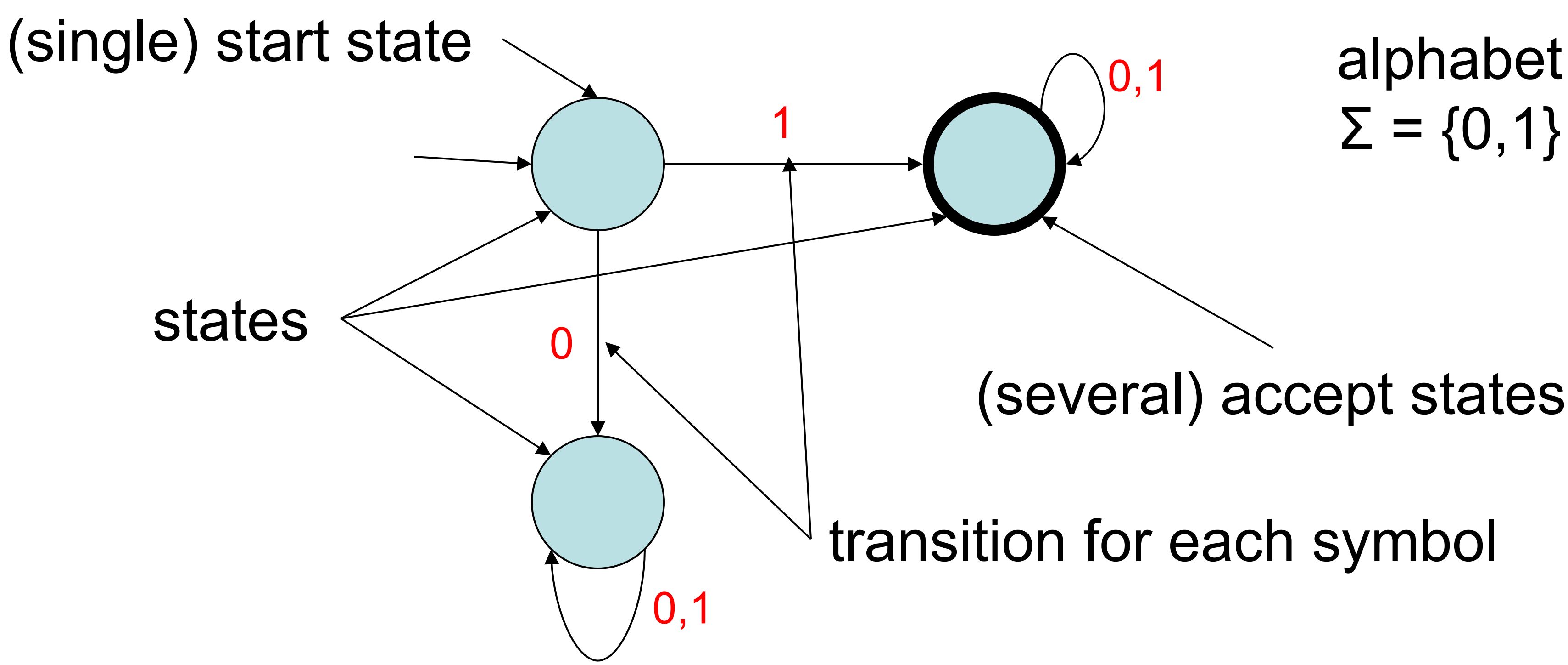
- Important ideas (continued):

A complete problem is a surrogate for the entire class.

Summary

Part I: automata

Finite Automata



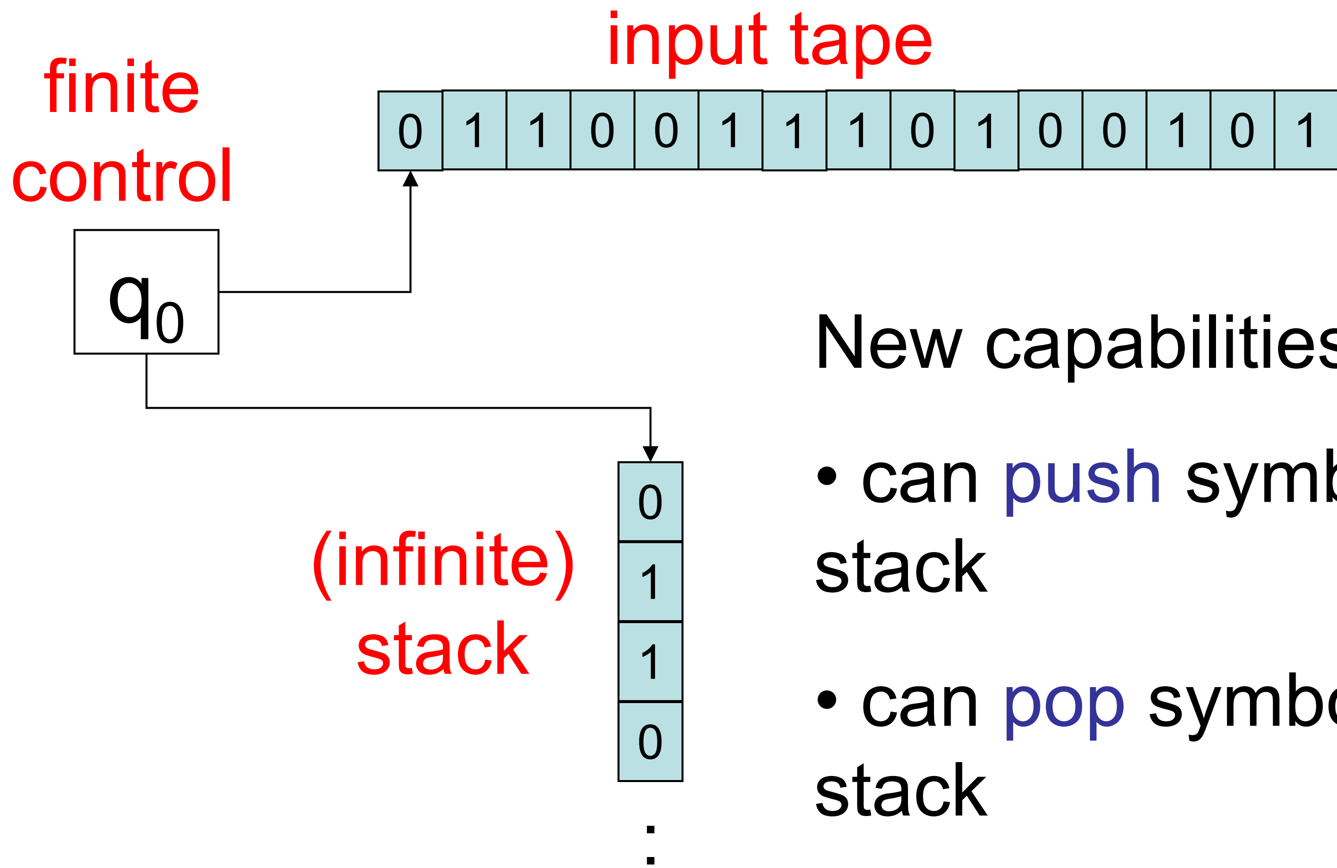
- read input one symbol at a time; follow arrows; accept if end in accept state

Finite Automata

- Non-deterministic variant: NFA
- Regular expressions built up from:
  - unions
  - concatenations
  - star operations

Main results: same set of languages recognized by FA, NFA and regular expressions ("regular languages").

Pushdown Automata



New capabilities:

- can push symbol onto stack
- can pop symbol off of stack

Context-Free Grammars

start symbol

terminal symbols

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

non-terminal symbols

production

Pushdown Automata

Main results: same set of languages recognized by NPDA, and context-free grammars ("context-free languages").

- and DPDA's weaker than NPDA's…

Non-regular languages

Pumping Lemma: Let L be a regular language. There exists an integer p ("pumping length") for which every $w \in L$ with $|w| \geq p$ can be written as

$w = xyz$ such that

1. for every $i \geq 0$, $xy^iz \in L$, and
2. $|y| > 0$, and
3. $|xy| \leq p$.

Pumping Lemma for CFLs

CFL Pumping Lemma: Let L be a CFL.

There exists an integer p ("pumping length") for which every $w \in L$ with $|w| \geq p$ can be written as

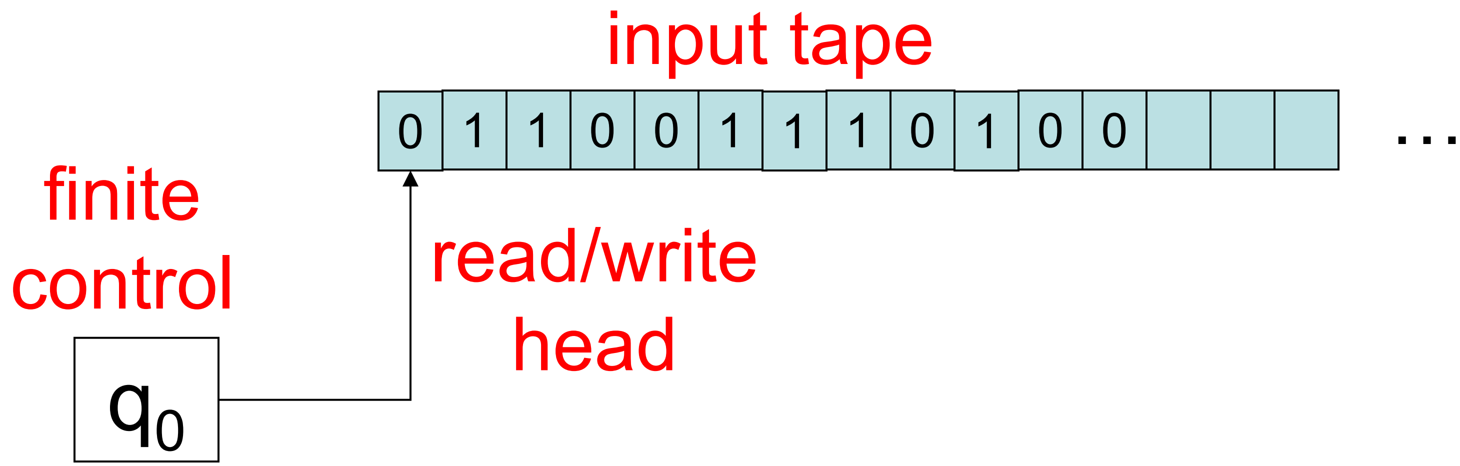
$$w = uvxyz \quad \text{such that}$$

1. for every $i \geq 0$, $uv^ixy^iz \in L$, and
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Summary

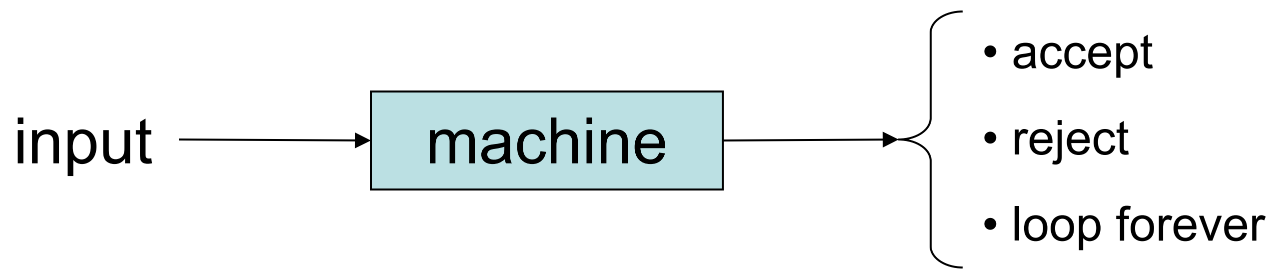
Part II: Turing Machines and decidability

Turing Machines



- New capabilities:
  - infinite tape
  - can read OR write to tape
  - read/write head can move left and right

Deciding and Recognizing



- TM M:
  - L(M) is the language it recognizes
  - if M rejects every x $\notin$ L(M) it decides L
  - set of languages recognized by some TM is called Turing-recognizable or recursively enumerable (RE)
  - set of languages decided by some TM is called Turing-decidable or decidable or recursive

Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an algorithm is:

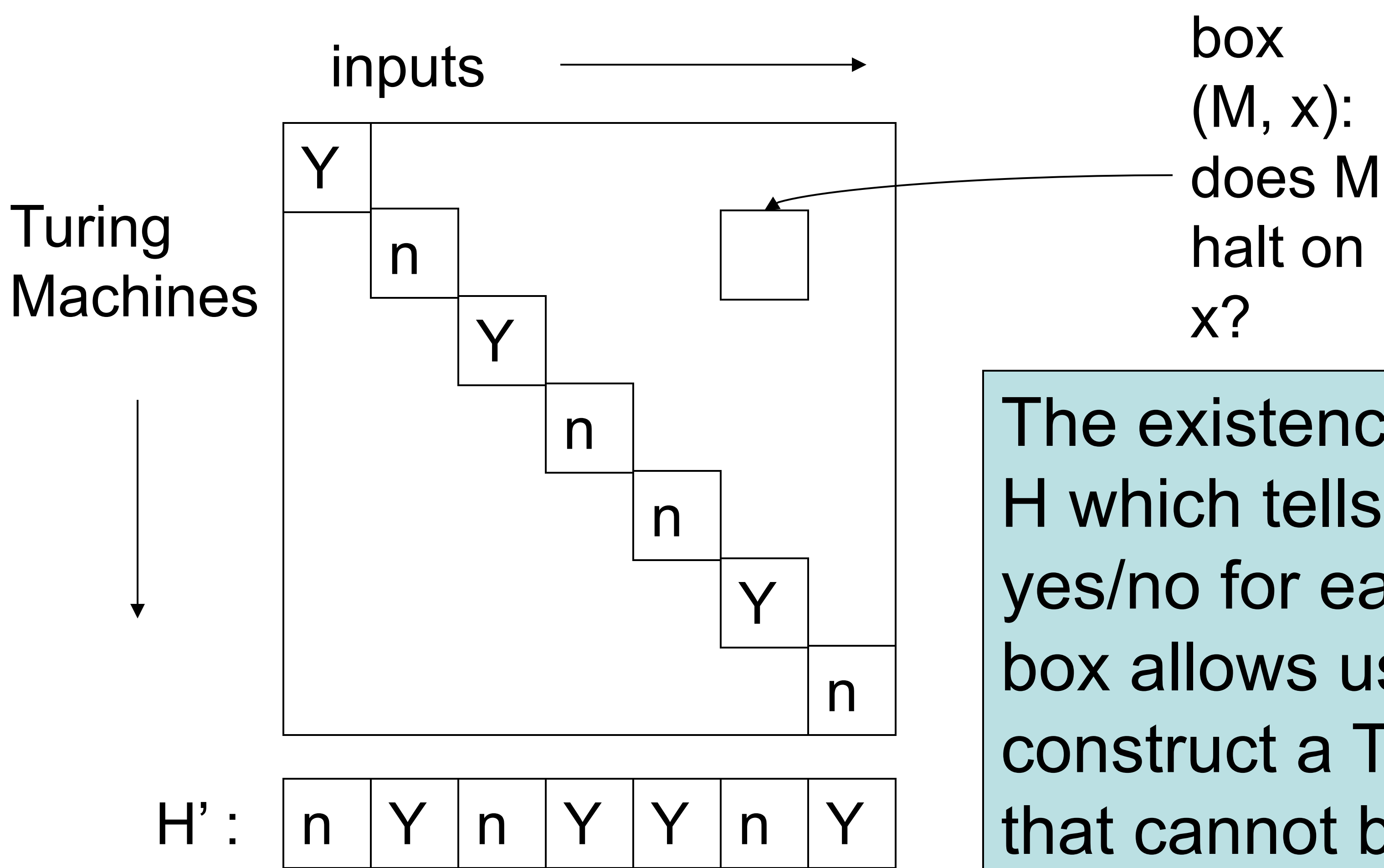
<u>The Church-Turing Thesis</u>

everything we can compute on a physical computer

can be computed on a Turing Machine

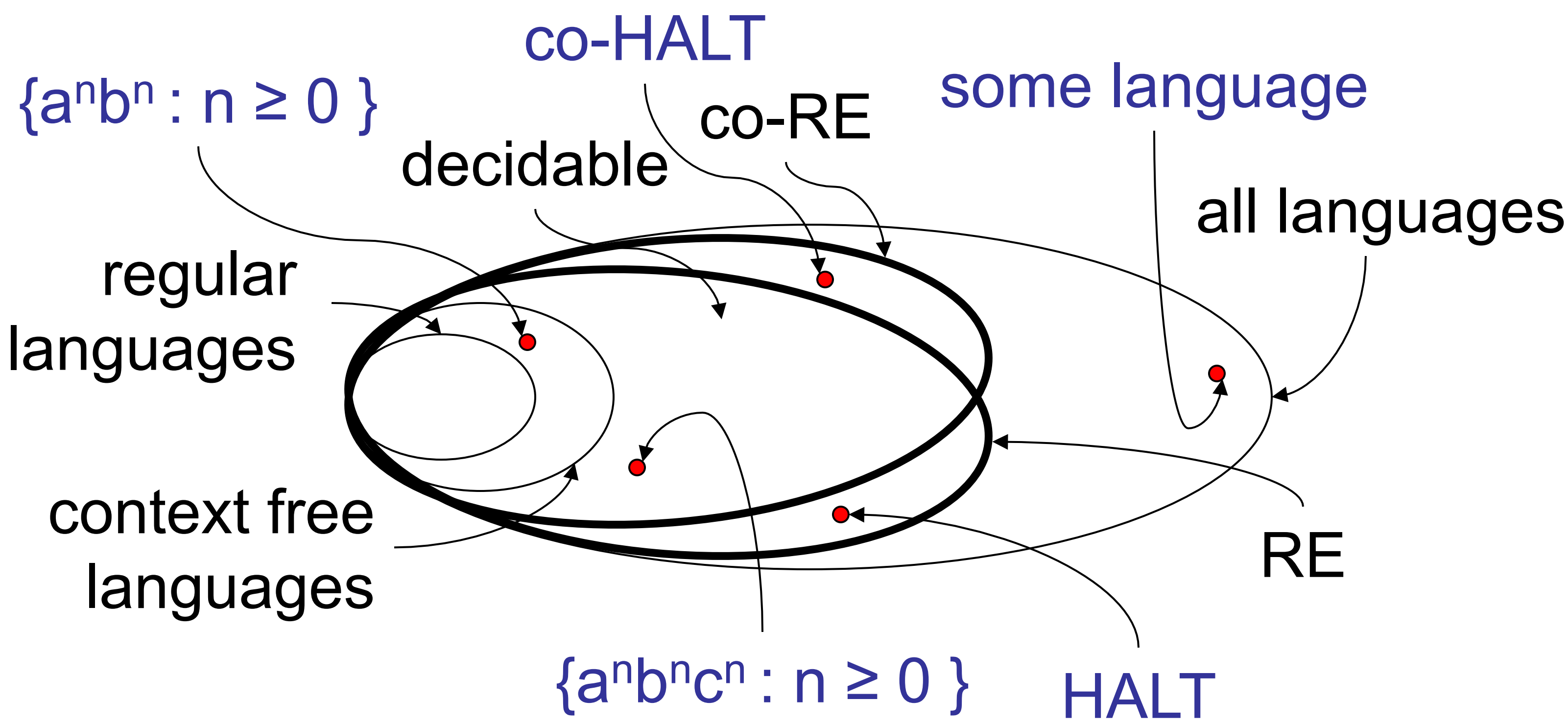
- Note: this is a belief, not a theorem.

The Halting Problem



The existence of H which tells us yes/no for each box allows us to construct a TM H’ that cannot be in the table.

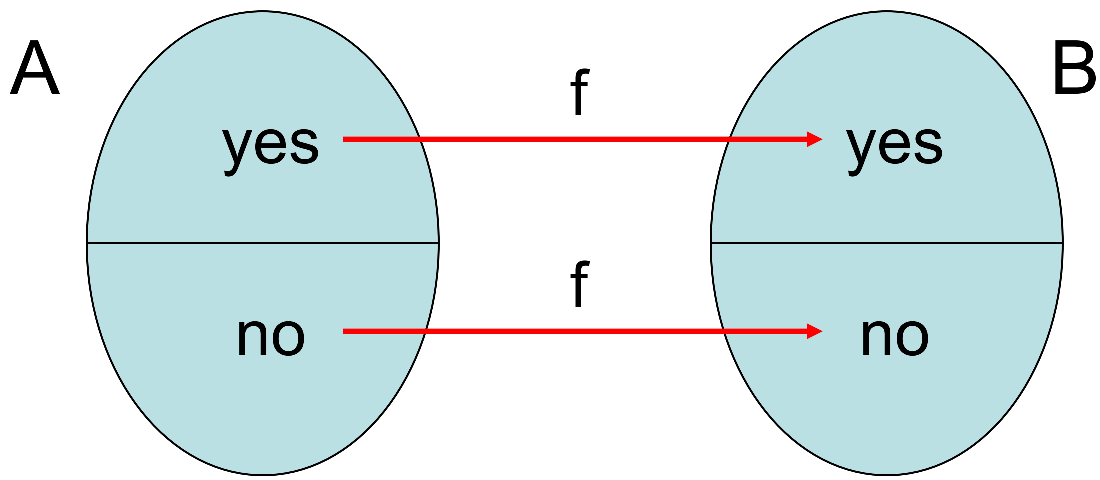
Decidable, RE, coRE…



some problems (e.g HALT) have no algorithms

Definition of reduction

- More refined notion of reduction:
  - “many-one” reduction (commonly)
  - “mapping” reduction (book)



reduction from language A to language B

Using reductions

- Used reductions to prove lots of problems were:
  - undecidable (reduce from undecidable)
  - non-RE (reduce from non-RE)
    - or show undecidable, and coRE
  - non-coRE (reduce from non-coRE)
    - or show undecidable, and RE

Rice's Theorem: Every nontrivial TM property is undecidable.

The Recursion Theorem

Theorem: Let T be a TM that computes fn:

$$t: \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$$

There is a TM R that computes the fn:

$$r: \Sigma^* \rightarrow \Sigma^*$$

defined as $r(w) = t(w, \langle R \rangle)$.

- In the course of computation, a Turing Machine can output its own description.

Incompleteness Theorem

<u>Theorem</u>: Peano Arithmetic is not complete.

(same holds for any reasonable proof system for number theory)

Proof outline:

– the set of theorems of PA is RE

– the set of true sentences (= Th(**N**)) is not RE

Summary

Part III: Complexity

Complexity

- Complexity Theory = study of what is computationally feasible (or tractable) with limited resources:
  - running *time*
  - storage *space*
  - number of *random bits*
  - degree of *parallelism*
  - rounds of *interaction*
  - *others…*

main focus

not in this course

Time and Space Complexity

<u>Definition</u>: the time complexity of a TM M is a function $f:\mathbf{N} \rightarrow \mathbf{N}$, where f(n) is the maximum number of steps M uses on any input of length n.

<u>Definition</u>: the space complexity of a TM M is a function $f:\mathbf{N} \rightarrow \mathbf{N}$, where f(n) is the maximum number of tape cells M scans on any input of length n.

Complexity Classes

Definition: $\mathrm{TIME}(t(n)) = \{L$: there exists a TM M that decides L in time $O(t(n))\}$

$$P = \cup_{k \geq 1} \mathrm{TIME}(n^k)$$

$$\mathrm{EXP} = \cup_{k \geq 1} \mathrm{TIME}(2^{n^k})$$

Definition: $\mathrm{SPACE}(t(n)) = \{L$: there exists a TM M that decides L in space $O(t(n))\}$

$$\mathrm{PSPACE} = \cup_{k \geq 1} \mathrm{SPACE}(n^k)$$

Complexity Classes

Definition: NTIME(t(n)) = {L : there exists a NTM M that decides L in time O(t(n))}

$$NP = \cup_{k \geq 1} NTIME(n^k)$$

- Theorem: $P \neq EXP$
- $P \subset NP \subset PSPACE \subset EXP$
- Don't know if any of the containments are proper.

Alternate definition of NP

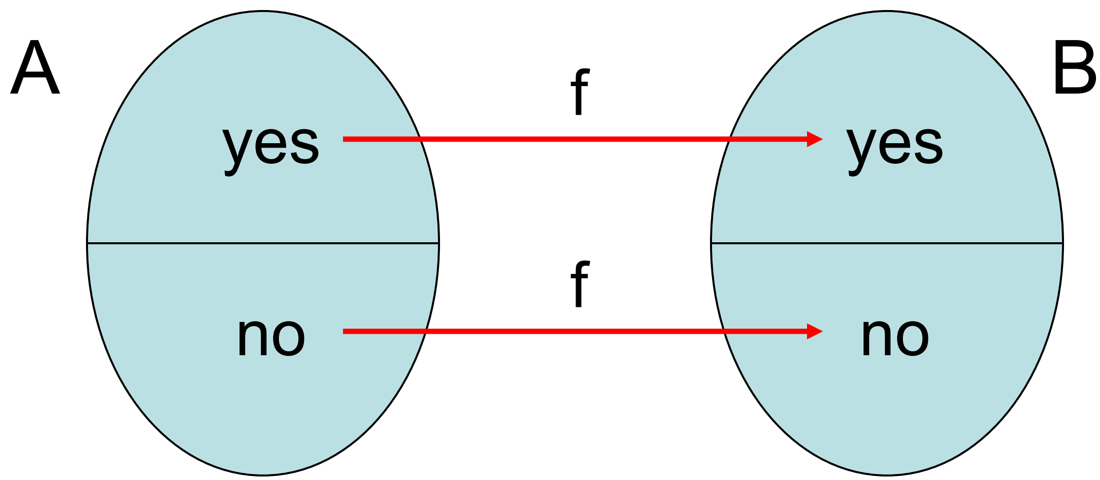
Theorem: language L is in NP if and only if it is expressible as:

$$L = \{ x \mid \exists\ y, |y| \leq |x|^k, (x, y) \in R \}$$

where R is a language in P.

Poly-time reductions

- Type of reduction we will use:
  - "many-one" poly-time reduction (commonly)
  - "mapping" poly-time reduction (book)



1. f poly-time computable
2. YES maps to YES
3. NO maps to NO

Hardness and completeness

Definition: a language L is C-hard if for every language $A \in C$, A poly-time reduces to L; i.e., $A \leq_P L$.

can show L is C-hard by reducing from a known C-hard problem

Definition: a language L is C-complete if L is C-hard and $L \in C$

Complete problems

- EXP-complete: ATM_B = {<M, x, m> : M is a TM that accepts x within at most m steps}
- PSPACE-complete: QSAT = {φ : φ is a 3-CNF, and $\exists x_1 \forall x_2 \exists x_3 \ldots \forall x_n \, \varphi(x_1, x_2, \ldots x_n)$}

- NP-complete: 3SAT = {φ : φ is a satisfiable 3-CNF formula}

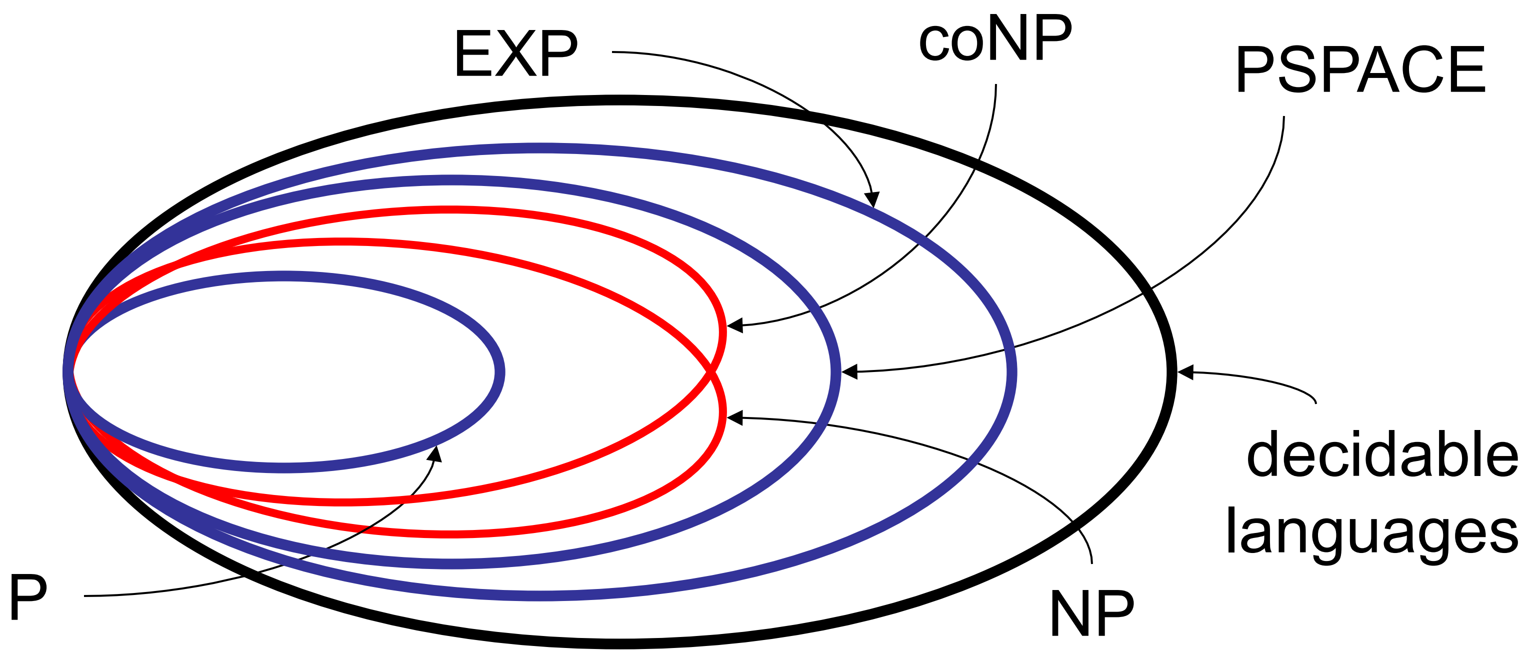
Lots of NP-complete problems

- Indendent Set
- Vertex Cover
- Clique
- Hamilton Path (directed and undirected)
- Hamilton Cycle and TSP
- Subset Sum
- NAE3SAT
- Max Cut
- Problem sets: max/min Bisection, 3-coloring, subgraph isomorphism, subset sum, (3,3)-SAT, Partition, Knapsack, Max2SAT…

Other complexity classes

- coNP – complement of NP
  - complete problems: UNSAT, DNF TAUTOLOGY
- NP intersect coNP
  - contains (decision version of) FACTORING
- PSPACE
  - complete problems: QSAT, GEOGRAPHY

Complexity classes



all containments believed to be proper