CS21
Decidability and Tractability

Lecture 22

February 28, 2018

Outline

- The complexity class PSPACE
  - a PSPACE-complete problem
  - PSPACE and 2-player games
- challenges to the extended Church-Turing Thesis
  - randomized computation
  - quantum computation

Space complexity

Definition: the space complexity of a TM M is a function

$$f: \mathbf{N} \rightarrow \mathbf{N}$$

where f(n) is the maximum number of tape cells M scans on any input of length n.

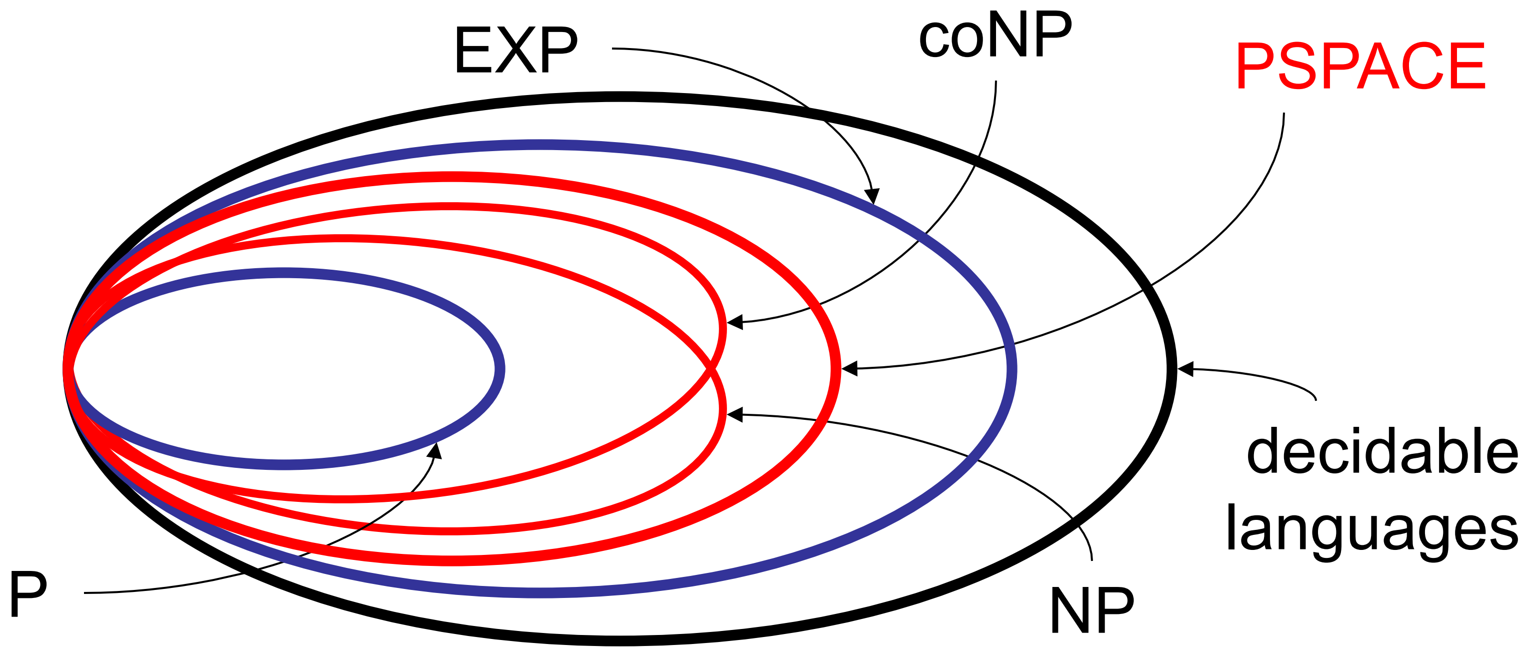
- "M uses space f(n)," "M is a f(n) space TM"

Space complexity

Definition: SPACE(t(n)) = {L : there exists a TM M that decides L in space O(t(n))}

$$\text{PSPACE} = \cup_{k \geq 1} \text{SPACE}(n^k)$$

PSPACE



- NP ⊂ PSPACE, coNP ⊂ PSPACE (proof?)
- PSPACE ⊂ EXP (proof?)
- containments believed to be proper

PSPACE

- A PSPACE-complete problem:
- Quantified Satisfiability:

$$\text{QSAT} = \{\varphi : \varphi \text{ is a 3-CNF, and } \exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \ldots \forall x_n\, \varphi(x_1, x_2, x_3, \ldots x_n)\}$$

- example: $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3)$

  $\exists x_1 \forall x_2 \exists x_3 \varphi$?

  YES: x_1=T; if x_2=T, set x_3=F; if x_2=F, set x_3=T

PSPACE

- A PSPACE-complete problem:
- Quantified Satisfiability:

$$\text{QSAT} = \{\varphi : \varphi \text{ is a 3-CNF, and}$$
$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \ldots \forall x_n\, \varphi(x_1, x_2, x_3, \ldots x_n)\ \}$$

- example: $\varphi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2)$

$\exists x_1 \forall x_2 \exists x_3 \varphi$?

NO: x_1=T; if x_2=T…; x_1=F; if x_2=T…

QSAT is **PSPACE**-complete

<u>Theorem</u>: QSAT is **PSPACE**-complete.

- Proof:
  - in **PSPACE:** $\exists x_1 \forall x_2 \exists x_3 \ldots Qx_n\, \varphi(x_1, x_2, \ldots, x_n)$?
  - "$\exists x_1$": for both $x_1 = 0$, $x_1 = 1$, recursively solve

    $\forall x_2 \exists x_3 \ldots Qx_n\, \varphi(x_1, x_2, \ldots, x_n)$?
    - if at least one "yes", return "yes"; else return "no"
  - "$\forall x_1$": for both $x_1 = 0$, $x_1 = 1$, recursively solve

    $\exists x_2 \forall x_3 \ldots Qx_n\, \varphi(x_1, x_2, \ldots, x_n)$?
    - if at least one "no", return "no"; else return "yes"
  - base case: evaluating a 3-CNF expression
  - poly(n) recursion depth
  - poly(n) bits of state at each level

QSAT is **PSPACE**-complete

- given TM M deciding L $\in$ **PSPACE**; input x
- 2^{n^k} possible configurations
- single START configuration
- assume single ACCEPT configuration

- define:

  REACH(X, Y, i) $\Leftrightarrow$ configuration Y reachable from configuration X in at most 2^i steps.

QSAT is **PSPACE**-complete

REACH(X, Y, i) ⇔ configuration Y reachable from configuration X in at most 2^i steps.

– Goal: produce 3-CNF $\varphi(w_1,w_2,w_3,\ldots,w_m)$ such that

$$\exists w_1 \forall w_2 \ldots \exists w_m\, \varphi(w_1,\ldots,w_m)$$
$$\Leftrightarrow \text{REACH(START, ACCEPT, } n^k)$$

QSAT is **PSPACE**-complete

- for i = 0, 1, … n^k produce quantified Boolean **expressions** $\psi_i(A, B, W)$ such that $\forall A,B$:

  $$\exists w_1 \forall w_2 \ldots \psi_i(A, B, W) \Leftrightarrow \text{REACH}(A, B, i)$$

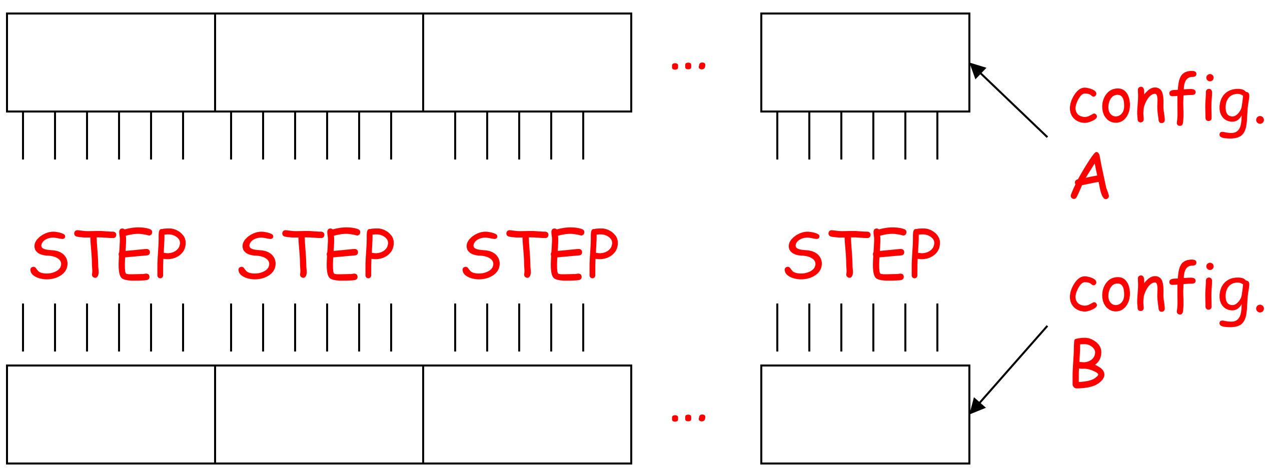
- convert ψ_{n^k} to 3-CNF φ
  - add variables V
- hardwire A = START, B = ACCEPT

  $$\exists w_1 \forall w_2 \ldots \exists V\ \varphi(W, V) \Leftrightarrow x \in L$$

QSAT is **PSPACE**-complete

– $\psi_o(A, B)$ = true iff
  - A = B or
  - A yields B in one step of M

} Boolean expression of size $O(n^k)$



QSAT is **PSPACE**-complete

– Key observation:

$$\text{REACH}(A, B, i+1)$$

$$\Leftrightarrow$$

$$\exists\, Z\big[\text{REACH}(A, Z, i) \wedge \text{REACH}(Z, B, i)\big]$$

– cannot define $\psi_{i+1}(A; B; Z, W, W')$ to be

$\exists\, Z\, [\exists w_1 \forall w_2 \ldots\, \psi_i(A, Z, W) \wedge \exists w_1' \,\forall w_2' \ldots\, \psi_i(Z, B, W')\,]$

(why?)

QSAT is **PSPACE**-complete

- Key idea: use quantifiers
- couldn't do $\psi_{i+1}(A; B; Z, W, W') =$

$$\exists Z\, [\exists w_1 \forall w_2 \ldots \psi_i(A, Z, W) \wedge \exists w_1' \, \forall w_2' \ldots \psi_i(Z, B, W')\,]$$

- define $\psi_{i+1}(A; B; Z, X, Y, W)$ to be

$$\exists Z \forall X \forall Y [((X=A \wedge Y=Z) \vee (X=Z \wedge Y=B)) \Rightarrow$$
$$\exists w_1 \forall w_2 \ldots \psi_i(X, Y, W)]$$

- $\psi_i(X, Y, W)$ is preceded by quantifiers
- move to front (they don't involve X,Y,Z,A,B)

QSAT is **PSPACE**-complete

$\psi_o(A, B)$ = true iff A = B or A yields B in 1 step

$\psi_{i+1}(A; B; Z, X, Y, W) =$

$\exists Z\ \forall X\ \forall Y[((X{=}A \wedge Y{=}Z) \vee (X{=}Z \wedge Y{=}B)) \Rightarrow \exists w_1 \forall w_2 \ldots\ \psi_i(X, Y, W)]$

– $|\psi_0| = O(n^k)$

– $|\psi_{i+1}| = O(n^k) + |\psi_i|$

– total size of ψ_{nk} is $O(n^k)^2 = \text{poly}(n)$

– reduction runs in polynomial time

PSPACE and games

QSAT = {φ : φ is a 3-CNF, and $\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \ldots \forall x_n\, \varphi(x_1, x_2, x_3, \ldots x_n)$ }

- Think of as 2-player game (player 1 trying to satisfy φ; player 2 adversary):
  - player 1 picks truth value for x_1
  - player 2 picks truth value for x_2
  - player 1 picks truth value for x_3…
- $\varphi \in$ QSAT iff player 1 can win no matter what player 2 does.