Outline

- NP-complete problems: independent set, vertex cover, clique
- NP-complete problems: Hamilton path and cycle, Traveling Salesperson Problem
- NP-complete problems: Subset Sum
- NP-complete problems: NAE-3-SAT, max cut

Vertex cover

- Definition: given a graph $G = (V, E)$, a vertex cover in $G$ is a subset $V' \subseteq V$ such that for all $(u, w) \in E$, $u \in V'$ or $w \in V'$
- A search problem:
  - given $G$, find the smallest vertex cover
- corresponding language (decision problem):

  $VC = \{(G, k) : G$ has a VC of size $\leq k\}$.

Vertex Cover is NP-complete

Theorem: the following language is NP-complete:

$VC = \{(G, k) : G$ has a VC of size $\leq k\}$.

- Proof:
  - Part 1: $VC \in$ NP. Proof?
  - Part 2: $VC$ is NP-hard.
    - reduce from?

Vertex Cover is NP-complete

- We are reducing from the language:

  $IS = \{(G, k) : G$ has an IS of size $\geq k\}$

to the language:

  $VC = \{(G, k) : G$ has a VC of size $\leq k\}$.
Vertex Cover is NP-complete

- How are IS, VC related?

- Given a graph $G = (V, E)$ with $n$ nodes
  - if $V' \subseteq V$ is a vertex cover of size $k$
  - then $V-V'$ is an independent set of size $n-k$

- Proof:
  - suppose not. Then there is some edge with both endpoints in $V-V'$. But then neither endpoint is in $V'$. Contradiction.

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Clique

- Definition: given a graph $G = (V, E)$, a clique in $G$ is a subset $V' \subseteq V$ such that for all $u, v \in V'$, $(u, v) \in E$
- A search problem:
  - given $G$, find the largest clique
- corresponding language (decision problem):
  - $CLIQUE = \{(G, k) : G$ has a clique of size $\geq k\}$

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Clique is NP-complete

- We are reducing from the language:
  - $IS = \{(G, k) : G$ has an IS of size $\geq k\}$

to the language:
  - $CLIQUE = \{(G, k) : G$ has a CLIQUE of size $\geq k\}$

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Clique is NP-complete

- The reduction:
  - given an instance of IS: $(G, k)$ f produces the pair $(G, n-k)$
  - f poly-time computable?
  - YES maps to YES?
    - IS of size $\geq k$ in $G$ $\Rightarrow$ VC of size $\leq n-k$ in $G$
  - NO maps to NO?
    - VC of size $\leq n-k$ in $G$ $\Rightarrow$ IS of size $\geq k$ in $G$

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Clique is NP-complete

- How are IS, CLIQUE related?

- Given a graph $G = (V, E)$, define its complement $G' = (V, E' = \{(u, v) : (u, v) \notin E\})$
  - if $V' \subseteq V$ is an independent set in $G$ of size $k$
  - then $V'$ is a clique in $G'$ of size $k$

- Proof:
  - Every pair of vertices $u, v \in V'$ has no edge between them in $G$. Therefore they have an edge between them in $G'$.
Clique is NP-complete

• How are IS, CLIQUE related?
• Given a graph \( G = (V, E) \), define its complement \( G' = (V, E' = \{(u,v) : (u,v) \notin E\}) \)
  – if \( V' \subseteq V \) is a clique in \( G' \) of size \( k \)
  – then \( V' \) is an independent set in \( G \) of size \( k \)
• Proof:
  – Every pair of vertices \( u,v \in V' \) has an edge between them in \( G' \). Therefore they have no edge between them in \( G \).

The reduction:
– given an instance of IS: \((G, k)\) f produces the pair \((G', k)\)
• f poly-time computable?
• YES maps to YES?
  – IS of size \( \geq k \) in \( G \) = CLIQUE of size \( \geq k \) in \( G' \)
• NO maps to NO?
  – CLIQUE of size \( \geq k \) in \( G' \) = IS of size \( \geq k \) in \( G \)

Hamilton Path

• Definition: given a directed graph \( G = (V, E) \), a Hamilton path in \( G \) is a directed path that touches every node exactly once.
• A language (decision problem):
  HAMPATH = \{ \((G, s, t) : G \) has a Hamilton path from \( s \) to \( t \) \}

Theorem: the following language is NP-complete:
HAMPATH = \{ \((G, s, t) : G \) has a Hamilton path from \( s \) to \( t \) \}
• Proof:
  – Part 1: HAMPATH \( \in \) NP. Proof?
  – Part 2: HAMPATH is NP-hard.
    • reduce from?

HAMPATH is NP-complete

• We are reducing from the language:
  3SAT = \{ \( \varphi : \varphi \) is a 3-CNF formula that has a satisfying assignment \}
  to the language:
  HAMPATH = \{ \((G, s, t) : G \) has a Hamilton path from \( s \) to \( t \) \}

• We want to construct a graph from \( \varphi \) with the following properties:
  – a satisfying assignment to \( \varphi \) translates into a Hamilton Path from \( s \) to \( t \)
  – a Hamilton Path from \( s \) to \( t \) can be translated into a satisfying assignment for \( \varphi \)
• We will build the graph up from pieces called gadgets that “simulate” the clauses and variables of \( \varphi \).
HAMPATH is NP-complete

- The variable gadget (one for each $x_i$):

  $x_i$ true:

  $x_i$ false:

• How to ensure that all $k$ clauses are satisfied?
  - need to add nodes
    - can be visited in path if the clause is satisfied
    - if visited in path, implies clause is satisfied by the assignment given by path through variable gadgets

• One clause gadget for each of $k$ clauses:

  for clause 1

  for clause 2
HAMPATH is NP-complete

\[ \varphi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_4 \lor x_5) \ldots \]

Case 1 (positive occurrence of \( v \) in clause):

- path must visit \( y \)
- must enter from \( x \), \( z \), or \( c \)
- must exit to \( z \) (\( x \) is taken)
- \( x \), \( c \) are taken
  can’t happen

Case 2 (negative occurrence of \( v \) in clause):

- path must visit \( y \)
- must enter from \( x \) or \( z \)
- must exit to \( z \) (\( x \) is taken)
- \( x \) is taken, can’t happen

What can go wrong?

- path has “intended form” unless return from clause gadget to different variable gadget

we will argue that this cannot happen: