CS21
Decidability and Tractability

Lecture 18

February 16, 2018

Outline

- the complexity class NP
  - 3-SAT is NP-complete
  - NP-complete problems: independent set, vertex cover, clique
  - NP-complete problems: Hamilton path and cycle,Traveling Salesperson Problem

Cook-Levin Theorem

- Gateway to proving lots of natural, important problems NP-complete is:

Theorem (Cook, Levin): 3SAT is NP-complete.

- Recall: 3SAT = {φ : φ is a CNF formula with 3 literals per clause for which there exists a satisfying truth assignment}

Cook-Levin Theorem

- Proof outline
  - show CIRCUIT-SAT is NP-complete

    CIRCUIT-SAT = {C : C is a Boolean circuit for which there exists a satisfying truth assignment}

  - show 3SAT is NP-complete (reduce from CIRCUIT SAT)

3SAT is NP-complete

Theorem: 3SAT is NP-complete

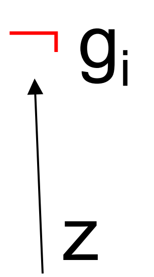
3SAT = {φ : φ is a 3-CNF formula for which there exists a satisfying truth assignment}

Proof:

- Part 1: need to show 3-SAT $\in$ NP
  - already done
- Part 2: need to show 3-SAT is NP-hard
  - we will give a poly-time reduction from CIRCUIT-SAT to 3-SAT

3SAT is NP-complete

- given a circuit C
  - variables $x_1, x_2, \ldots, x_n$
  - AND ($\wedge$), OR ($\vee$), NOT ($\neg$) gates $g_1, g_2, \ldots, g_m$
- reduction f(C) produces these clauses for φ on variables $x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_m$:



$$\left.\begin{array}{l} \bullet\ (g_i \vee z) \\ \bullet\ (\neg z \vee \neg g_i) \end{array}\right\} (z \Leftrightarrow \neg g_i)$$

3SAT is NP-complete

- given a circuit C
  - variables $x_1, x_2, \ldots, x_n$
  - AND ($\wedge$), OR ($\vee$), NOT ($\neg$) gates $g_1, g_2, \ldots, g_m$
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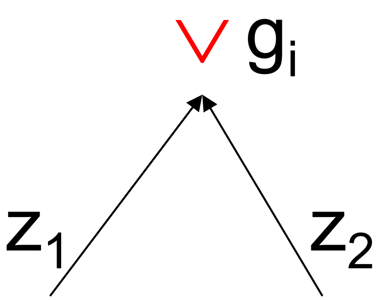


- $(\neg z_1 \vee g_i)$
- $(\neg z_2 \vee g_i)$
- $(\neg g_i \vee z_1 \vee z_2)$

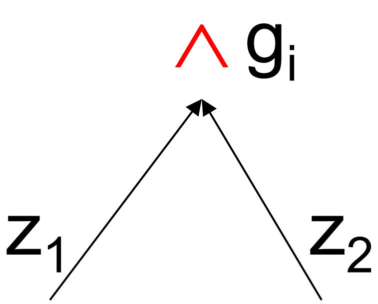
$(z_1 \vee z_2 \Leftrightarrow g_i)$

3SAT is NP-complete

– given a circuit C

- variables $x_1, x_2, \ldots, x_n$
- AND ($\wedge$), OR ($\vee$), NOT ($\neg$) gates $g_1, g_2, \ldots, g_m$

– reduction f(C) produces these clauses for φ on variables $x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_m$:



$$\left.\begin{array}{l} \bullet\ (\neg g_i \vee z_1) \\ \bullet\ (\neg g_i \vee z_2) \\ \bullet\ (\neg z_1 \vee \neg z_2 \vee g_i) \end{array}\right\} \quad (z_1 \wedge z_2 \Leftrightarrow g_i)$$

3SAT is NP-complete

- finally, reduction f(C) produces single clause (g_m) where g_m is the output gate.
- f(C) computable in poly-time?
  - yes, simple transformation
- YES maps to YES?
  - if C(x) = 1, then assigning x-values to x-variables of φ and gate values of C when evaluating x to the g-variables of φ gives satsifying assignment.

3SAT is NP-complete

- NO maps to NO?
  - show that φ satisfiable implies C satisfiable
  - satisfying assignment to φ assigns values to x-variables and g-variables
  - output gate g_m must be assigned 1
  - every other gate must be assigned value it would take given values of its inputs.
  - the assignment to the x-variables must be a satisfying assignment for C.

Search vs. Decision

- Definition: given a graph G = (V, E), an independent set in G is a subset $V' \subseteq V$ such that for all $u,w \in V'$ $(u,w) \notin E$
- A problem:

  given G, find the largest independent set
- This is called a search problem
  - searching for *optimal* object of some type
  - comes up frequently

Search vs. Decision

- We want to talk about languages (or decision problems)
- Most search problems have a natural, related decision problem by adding a bound "k"; for example:
  - search problem: given G, find the largest independent set
  - decision problem: given (G, k), is there an independent set of size *at least* k

Ind. Set is NP-complete

Theorem: the following language is NP-complete:

$$IS = \{(G, k) : G \text{ has an IS of size} \geq k\}.$$

- Proof:
  - Part 1: IS $\in$ NP. Proof?
  - Part 2: IS is NP-hard.
    - reduce from 3-SAT

Ind. Set is NP-complete

- We are reducing from the language:

  3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

  to the language:

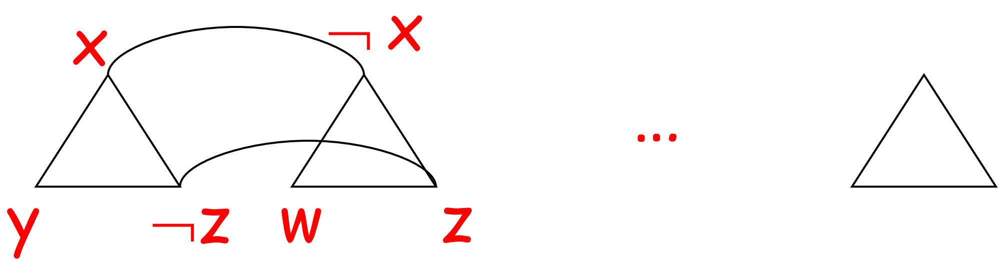
  IS = {(G, k) : G has an IS of size $\geq$ k}.

Ind. Set is NP-complete

The reduction f: given

$$\varphi = (x \lor y \lor \neg z) \land (\neg x \lor w \lor z) \land \ldots \land (\ldots)$$

we produce graph G_φ:

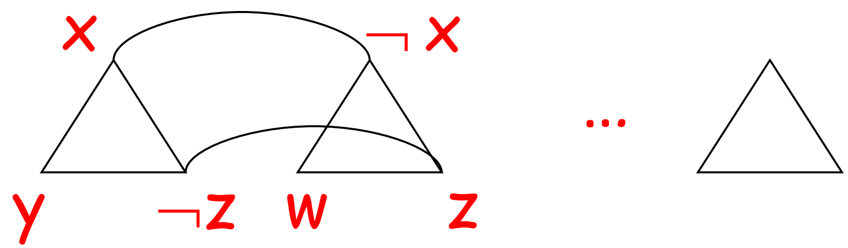


- one triangle for each of m clauses
- edge between every pair of contradictory literals
- set k = m

Ind. Set is NP-complete

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

$f(\varphi) =$
(G, # clauses)

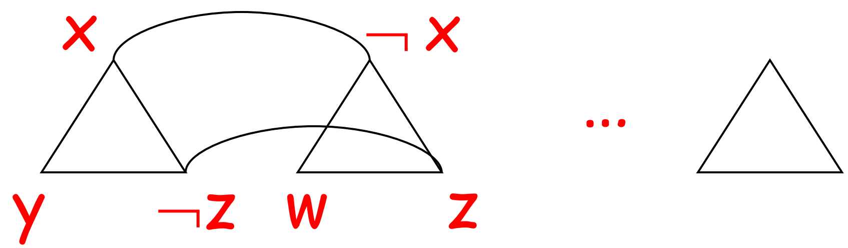


- Is f poly-time computable?
- YES maps to YES?
  - 1 true literal per clause in satisfying assign. A
  - choose corresponding vertices (1 per triangle)
  - IS, since no contradictory literals in A

Ind. Set is NP-complete

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

$f(\varphi) =$
(G, # clauses)



- NO maps to NO?
  - IS can have at most 1 vertex per triangle
  - IS of size ≥ # clauses must have exactly 1 per
  - since IS, no contradictory vertices
  - can produce satisfying assignment by setting these literals to true

Vertex cover

- Definition: given a graph $G = (V, E)$, a vertex cover in G is a subset $V' \subseteq V$ such that for all $(u,w) \in E$, $u \in V'$ or $w \in V'$
- A search problem:

  given G, find the smallest vertex cover

- corresponding language (decision problem):

$$VC = \{(G, k) : G \text{ has a VC of size} \leq k\}.$$

Vertex Cover is NP-complete

Theorem: the following language is NP-complete:

$VC = \{(G, k) : G \text{ has a VC of size} \leq k\}$.

- Proof:
  - Part 1: VC $\in$ NP. Proof?
  - Part 2: VC is NP-hard.
    - reduce from?

Vertex Cover is NP-complete

- We are reducing from the language:

$$IS = \{(G, k) : G \text{ has an IS of size} \geq k\}$$

to the language:

$$VC = \{(G, k) : G \text{ has a VC of size} \leq k\}.$$

Vertex Cover is NP-complete

- How are IS, VC related?

- Given a graph G = (V, E) with n nodes
  - if V' $\subseteq$ V is an independent set of size k
  - then V-V' is a vertex cover of size n-k
- Proof:
  - suppose not. Then there is some edge with neither endpoint in V-V'. But then both endpoints are in V'. contradiction.

Vertex Cover is NP-complete

- How are IS, VC related?

- Given a graph G = (V, E) with n nodes
  - if V' $\subseteq$ V is a vertex cover of size k
  - then V-V' is an independent set of size n-k
- Proof:
  - suppose not. Then there is some edge with both endpoints in V-V'. But then neither endpoint is in V'. contradiction.

Vertex Cover is NP-complete

The reduction:

- given an instance of IS: (G, k) f produces the pair (G, n-k)

- f poly-time computable?
- YES maps to YES?
  - IS of size $\geq$ k in G $\Rightarrow$ VC of size $\leq$ n-k in G
- NO maps to NO?
  - VC of size $\leq$ n-k in G $\Rightarrow$ IS of size $\geq$ k in G

Clique

- Definition: given a graph $G = (V, E)$, a clique in G is a subset $V' \subseteq V$ such that for all $u,v \in V'$, $(u, v) \in E$
- A search problem:

  given G, find the largest clique

- corresponding language (decision problem):

  CLIQUE = $\{(G, k) : G \text{ has a clique of size} \geq k\}$.

Clique is NP-complete

<u>Theorem</u>: the following language is NP-complete:

CLIQUE = {(G, k) : G has a clique of size $\geq$ k}

- Proof:
  - Part 1: CLIQUE $\in$ NP. Proof?
  - Part 2: CLIQUE is NP-hard.
    - reduce from?

Clique is NP-complete

- We are reducing from the language:

IS = {(G, k) : G has an IS of size $\geq$ k}

to the language:

CLIQUE = {(G, k) : G has a CLIQUE of size $\geq$ k}.

Clique is NP-complete

- How are IS, CLIQUE related?
- Given a graph G = (V, E), define its complement G' = (V, E' = {(u,v) : (u,v) $\notin$ E})
  - if V' $\subseteq$ V is an independent set in G of size k
  - then V' is a clique in G' of size k
- Proof:
  - *Every* pair of vertices u,v $\in$ V' has no edge between them in G. Therefore they have an edge between them in G'.

Clique is NP-complete

- How are IS, CLIQUE related?
- Given a graph G = (V, E), define its complement G' = (V, E' = {(u,v) : (u,v) $\notin$ E})
  - if V' $\subseteq$ V is a clique in G' of size k
  - then V' is an independent set in G of size k
- Proof:
  - *Every* pair of vertices u,v 2 V' has an edge between them in G'. Therefore they have no edge between them in G.

Clique is NP-complete

The reduction:

- given an instance of IS: (G, k) f produces the pair (G', k)

- f poly-time computable?
- YES maps to YES?
  - IS of size $\geq$ k in G $\Rightarrow$ CLIQUE of size $\geq$ k in G'
- NO maps to NO?
  - CLIQUE of size $\geq$ k in G' $\Rightarrow$ IS of size $\geq$ k in G

Hamilton Path

- Definition: given a directed graph G = (V, E), a Hamilton path in G is a directed path that touches every node exactly once.

- A language (decision problem):

  HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

HAMPATH is NP-complete

Theorem: the following language is NP-complete:

HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

- Proof:
  - Part 1: HAMPATH $\in$ NP. Proof?
  - Part 2: HAMPATH is NP-hard.
    - reduce from?

HAMPATH is NP-complete

- We are reducing from the language:

  3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

  to the language:

  HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}