CS21 Decidability and Tractability

Lecture 13

February 3, 2023

Outline

- reductions
- many-one reductions
- undecidable problems
  – computation histories
  – surprising contrasts between decidable/undecidable
- Rice's Theorem

Example reduction

- Preceding reduction proved:

Theorem: A_{TM} is undecidable.

Proof (recap):

- suppose A_{TM} is decidable
- we showed how to use A_{TM} to decide HALT
- conclude HALT is decidable. Contradiction.

Another example

- Try to prove undecidable:

$$E_{TM} = \{\langle M\rangle : L(M) = \emptyset\}$$

- which problem should we **reduce from**?
  - HALT = {<M, w> : M halts on input w}
  - A_{TM} = {<M, w> : M accepts input w}
- Some things we can do:
  - check if <M> $\in E_{TM}$
  - construct another TM M' and check if <M'> $\in E_{TM}$

Another example

- We are given input $\langle M, w \rangle$
- We want to use a procedure that decides E_{TM} to decide if $\langle M, w \rangle \in A_{TM}$

- Idea:
  - check if $\langle M \rangle \in E_{TM}$
  - if not?
  - helpful if could make M reject everything except possibly w.

Another example

Is this OK?
finite # of states?

- Construct TM M':
  - on input x, if $x \neq w$, then reject
  - else simulate M on x, and accept if M does.
- on input <M, w>
  - construct M' from description of M
  - check if $M' \in E_{TM}$
    - if no, M must accept w; **ACCEPT**
    - if yes, M cannot accept w; **REJECT**

Another example

- Preceding reduction proved:

<u>Theorem</u>: E_{TM} is undecidable.

Proof (recap):

- suppose E_{TM} is decidable
- we showed how to use E_{TM} to decide A_{TM}
- conclude A_{TM} is decidable. Contradiction.

Example reduction

- We proved

  $$A_{TM} = \{<M, w> : M \text{ accepts input } w\}$$

  undecidable, by reduction from

  $$HALT = \{<M, w> : M \text{ halts on input } w\}$$

- We proved

  $$E_{TM} = \{<M> : L(M) = \emptyset\}$$

  undecidable by reduction from A_{TM}

Definition of reduction

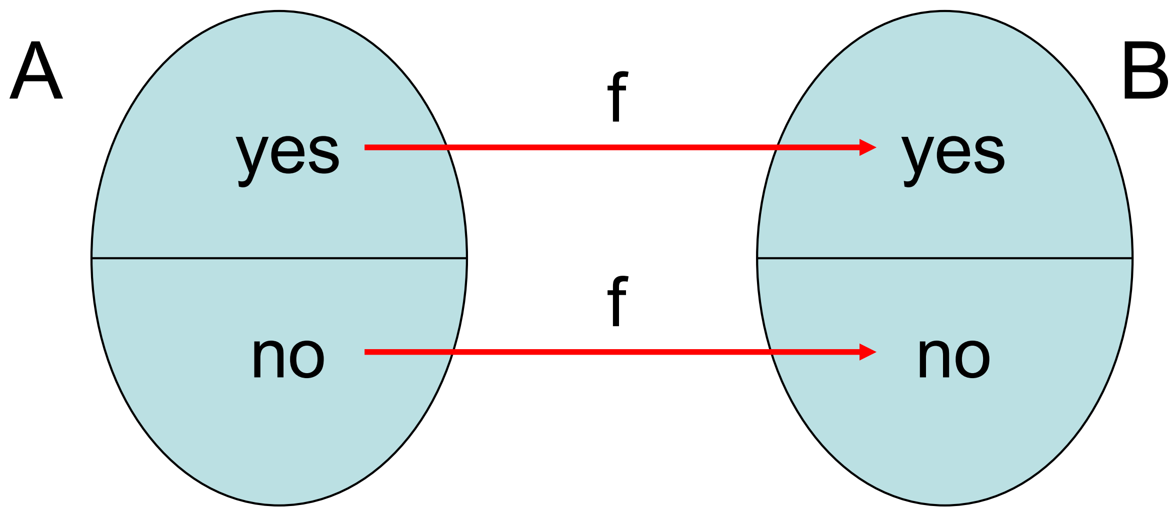
- Can you reduce co-HALT to HALT?

- We know that HALT is RE
- Does this show that co-HALT is RE?
  - recall, we showed co-HALT is not RE

- our current notion of reduction cannot distinguish complements

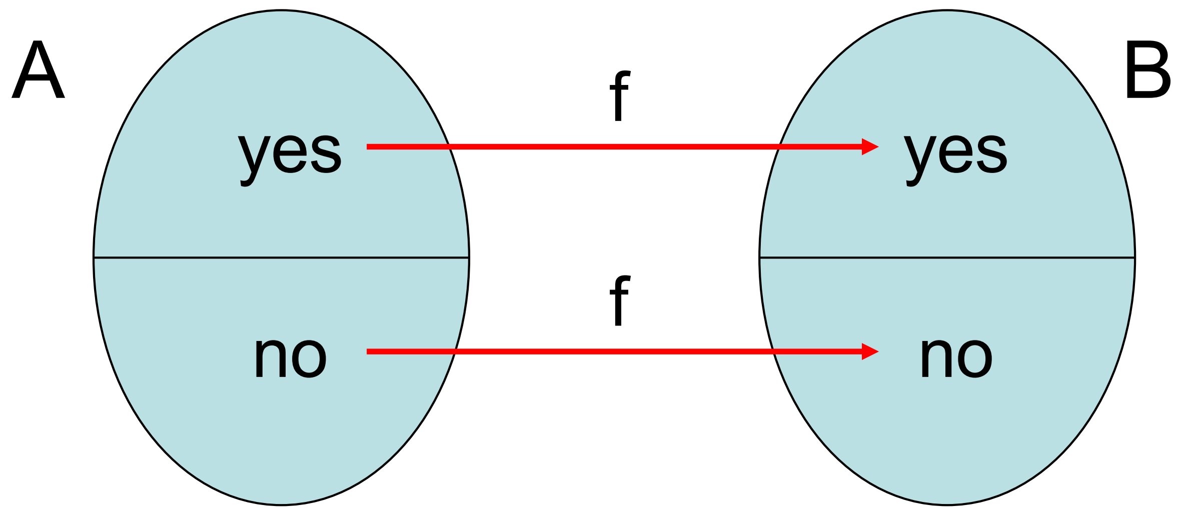
Definition of reduction

- More refined notion of reduction:
  - "many-one" reduction (commonly)
  - "mapping" reduction (book)



reduction from language A to language B

Definition of reduction



- function f should be computable

Definition: $f : \Sigma^* \rightarrow \Sigma^*$ is computable if there exists a TM M_f such that on every $w \in \Sigma^*$ M_f halts on w with $f(w)$ written on its tape.

Definition of reduction

- Notation: “A many-one reduces to B” is written

$$A \leq_m B$$

  – “yes maps to yes and no maps to no” means:

$w \in A$ maps to $f(w) \in B$ & $w \notin A$ maps to $f(w) \notin B$

- B is at least as “hard” as A
  – more accurate: B at least as “expressive” as A

Using reductions

Definition: $A \leq_m B$ if there is a computable function f such that for all w

$$w \in A \Leftrightarrow f(w) \in B$$

Theorem: if $A \leq_m B$ and B is decidable then A is decidable

Proof:

– decider for A: on input w, compute f(w), run decider for B, do whatever it does.

Using reductions

- Main use: given language NEW, prove it is undecidable by showing OLD $\leq_m$ NEW, where OLD known to be undecidable
  - proof by contradiction
  - if NEW decidable, then OLD decidable
  - OLD undecidable. Contradiction.
- common to reduce in wrong direction.
- review this argument to check yourself.

Using reductions

<u>Theorem</u>: if A $\leq_m$ B and B is RE then A is RE

Proof:

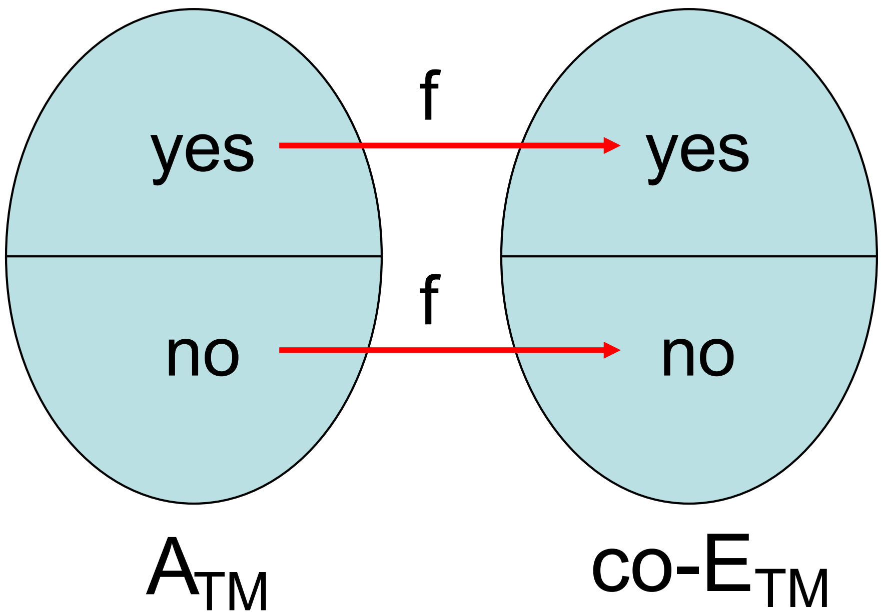
- TM for recognizing A: on input w, compute f(w), run TM that recognizes B, do whatever it does.

- Main use: given language NEW, prove it is not RE by showing OLD $\leq_m$ NEW, where OLD known to be not RE.

Many-one reduction example

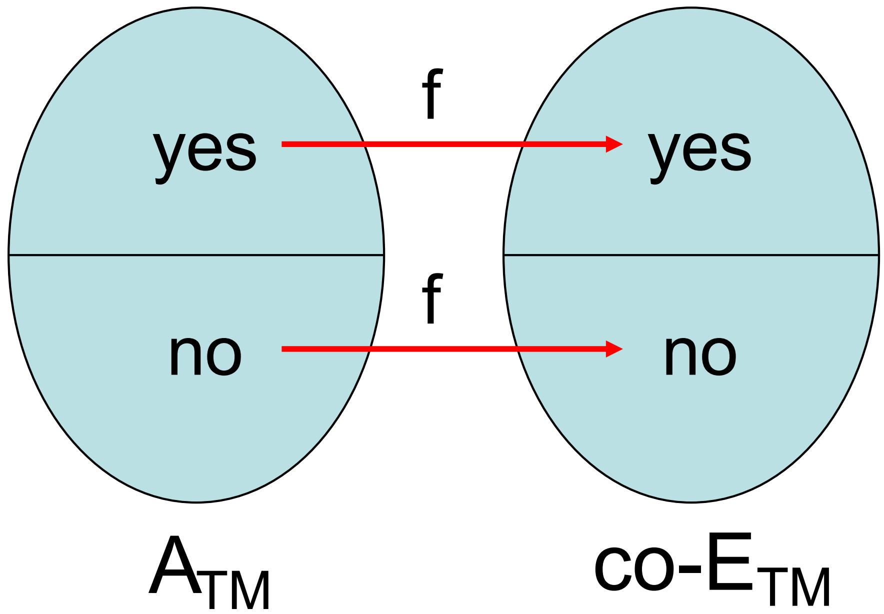
- Showed E_{TM} undecidable. Consider:

$$\text{co-}E_{TM} = \{\langle M\rangle : L(M) \neq \emptyset\}$$



- $f(\langle M, w\rangle) = \langle M'\rangle$ where M' is TM that
  - on input x, if $x \neq w$, then reject
  - else simulate M on x, and accept if M does
- f clearly computable

Many-one reduction example



- f(<M, w>) = <M'> where M' is TM that
  - on input x, if $x \neq w$, then reject
  - else simulate M on x, and accept if M does
- f clearly computable

- yes maps to yes?
  - if <M, w> $\in A_{TM}$ then f(M, w) $\in$ co-E_{TM}
- no maps to no?
  - if <M, w> $\notin A_{TM}$ then f(M, w) $\notin$ co-E_{TM}

Undecidable problems

Theorem: The language

REGULAR = {<M>: M is a TM and L(M) is regular}

is undecidable.

Proof:

- reduce from A_{TM} (i.e. show $A_{TM} \leq_m$ REGULAR)
- what should f(<M, w>) produce?

Undecidable problems

Proof:

– f(<M, w>) = <M'> described below

on input x:

• if x has form 0^n1^n, accept

• else simulate M on w and accept x if M accepts

• is f computable?

• YES maps to YES?

$\langle M, w\rangle \in A_{TM} \Rightarrow f(M, w) \in \text{REGULAR}$

• NO maps to NO?

$\langle M, w\rangle \notin A_{TM} \Rightarrow f(M, w) \notin \text{REGULAR}$