CS 21 Decidability and Tractability	**Winter 2018**

Final

Out: March 7 — *Due:* **March 14, noon**

This is the final exam. You may consult only the course notes and text (Sipser). You may not collaborate. There are 5 problems on two pages. Please attempt all problems. **To facilitate grading, please turn in each problem on a separate sheet of paper and put your name on each sheet. Do not staple the separate sheets.** Good luck!

Instructions for turning in the exam: Please turn in your exams to Diane Goodfellow in Annenberg 246 before noon on Wednesday March 14.

1. Consider the following 2-person game. The game is played on an $n \times n$ "board" with non-negative integers in each of the n^2 cells. There is a specified "target" value B which is also a nonnegative integer. The players take turns selecting integers from successive rows: player one selects an integer in row 1, then player two selects an integer in row 2, then player one selects an integer in row 3, etc... After an integer has been selected from the last row, the game ends. If the sum of all of the selected integers equals B, then player 1 wins; otherwise player 2 wins.

   Given a game board (a square matrix M of nonnegative integers) and a nonnegative integer B, we can ask whether there is a win for player one (i.e., player one can win no matter what player two does). Prove that the language L consisting of pairs $\langle M, B\rangle$ for which there is a win for player one is PSPACE-complete. In other words prove:

   (a) L is in PSPACE, and

   (b) L is PSPACE-hard. Here it may be useful to recall the two-player game interpretation of QSAT from Lectures 23-24.

2. Let L be the language over the alphabet $\Sigma = \{a, b, c\}$ consisting of exactly those strings with an *unequal* number of a's and b's (and any number of c's). Is L (i) regular, (ii) context-free but not regular, or (iii) not context free? Prove that your classification is correct.

3. For a language $L \subseteq \Sigma^*$ and a string $y \in \Sigma^*$, the language

   $$L_{-y} = \{xz : x \in \Sigma^* \text{ and } z \in \Sigma^* \text{ and } xyz \in L\}$$

   consists of all strings in L with the string y deleted from them.

   (a) Prove that if L is regular, then L_{-y} is regular. Hint: make $|y| + 1$ copies of a DFA recognizing L.

   (b) Prove that if L is R.E., then L_{-y} is R.E.

4. Is the following statement true or false? Give a complete proof of your assertion.

   "If for some fixed k, an NP-complete language has an $O(n^k)$-time algorithm, then every language in NP has an $O(n^k)$-time algorithm."

5. Each of the following languages is either in P, or it is NP-complete. **Choose 4 out of the 5 problems, and for each one, prove that it is NP-complete, or prove that it is in P. Please indicate clearly which 4 you are choosing, and provide solutions for only those 4.**

   For two of the problems below, you will need to recall that in a graph, the *degree* of a vertex v, denoted $d(v)$, is the number of edges that touch that vertex; the *maximum degree* of a graph is the maximum, over vertices v, of $d(v)$.

   (a) This problem is a variant of INDEPENDENT SET in bounded-degree graphs. The language in question is the set of all pairs (G, k) for which G is a graph with maximum degree at most 4 containing an independent set of size at least k.

   (b) This problem is a variant of UNDIRECTED HAMILTON PATH in bounded-degree graphs. The language in question is the set of all triples (G, s, t) for which G is an undirected graph with maximum degree at most 2 containing a Hamilton path from node s to node t.

   (c) Given a universe U and a collection of subsets $\mathcal{C} = \{S_1, S_2, S_3, \ldots, S_n\}$, with each $S_i \subseteq U$, we say that a subset $H \subseteq U$ is a *hitting set* if each S_i contains at least one element of H. In this problem we are interested in the case in which each S_i has size at most 2. The language in question is

   $$\text{HITTING SET-2} = \{(\mathcal{C}, k) \; : \; \text{for all } S_i \in \mathcal{C},\ |S_i| \leq 2, \text{ and there is a hitting set } H \subseteq U \text{ with } |H| \leq k\}.$$

   (d) The language consisting of 2-CNF formulas ϕ for which there exists an assignment that satisfies at least 3/4 of the first 1000 clauses, and all of the other clauses.

   (e) The language consisting of 2-CNF formulas ϕ for which there exists an assignment that satisfies all of the first 1000 clauses, and at least 3/4 of the other clauses.