1. Consider the following 2-person game. The game is played on a directed acyclic graph whose nodes are labeled with integers. There is a specified start-node $s$.

Starting from node $s$, the players take turns selecting an outgoing edge from the current node: player one selects an outgoing edge from nodes $s$, which takes them to a node $v$, then player two selects an outgoing edge from node $v$, which takes them to a node $u$, then player one selects an outgoing edge from node $u$, and so on. We keep a running sum of the integers encountered as this path from $s$ in the graph is traversed. The game ends when a sink (a node with no outgoing edges) is reached. At that point player one wins if the running sum equals zero; otherwise player two wins.

Given a game instance (a directed acyclic graph $G$ labeled with integers, and a start node $s$) we can ask whether there is a win for player one (i.e., player one can win no matter what player two does). Prove that the language $L$ consisting of those game instances for which there is a win for player one is PSPACE-complete. In other words prove:

(a) $L$ is in PSPACE, and
(b) $L$ is PSPACE-hard. Here it may be useful to recall the two-player game interpretation of QSAT from Lecture 24. Hint: in your reduction, it is sufficient for the graph to be a $layered$ graph, with at most 3 nodes per layer. A layered directed graph is one in which the nodes can be partitioned into subsets (“layers”) $V = L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_k$ and the only directed edges are edges between adjacent layers; i.e., from nodes in layer $L_i$ to nodes in layer $L_{i+1}$.

2. Let $L$ be the language over the alphabet $\Sigma = \{a, b, c\}$ consisting of exactly those strings with an $unequal$ number of $a$’s and $b$’s (and any number of $c$’s). Is $L$ (i) regular, (ii) context-free but not regular, or (iii) not context free? Prove that your classification is correct.

3. For a language $L \subseteq \Sigma^*$ and a string $w \in \Sigma^*$, the language

$$L_{-w} = \{xy : x \in \Sigma^* \text{ and } y \in \Sigma^* \text{ and } xwy \in L\}$$

consists of all strings in $L$ with the string $w$ deleted from them.
(a) Prove that if $L$ is regular, then $L - w$ is regular. Hint: make $|w| + 1$ copies of a DFA recognizing $L$.

(b) Prove that if $L$ is R.E., then $L - w$ is R.E.

4. Suppose someone says they wish to prove the following reasonable-sounding statement:

“If some NP-complete language has an $O(n^2)$-time algorithm, then every language in NP has an $O(n^4)$-time algorithm.”

Show that such a proof is not possible. In other words, prove that even if some NP-complete language had an $O(n^2)$-time algorithm, the conclusion (that every language in NP has an $O(n^4)$-time algorithm) is false.

5. Each of the following languages is either in P, or it is NP-complete. Choose 4 out of the 5 problems, and for each one, prove that it is NP-complete, or prove that it is in P. Please indicate clearly which 4 you are choosing, and provide solutions for only those 4.

For two of the problems below, you will need to recall that in a graph, the degree of a vertex $v$, denoted $d(v)$, is the number of edges that touch that vertex; the maximum degree of a graph is the maximum, over vertices $v$, of $d(v)$.

(a) This problem is a variant of INDEPENDENT SET in bounded-degree graphs. The language in question is the set of all pairs $(G, k)$ for which $G$ is a graph with maximum degree at most 4 containing an independent set of size at least $k$.

(b) This problem is a variant of UNDIRECTED HAMILTON PATH in bounded-degree graphs. The language in question is the set of all triples $(G, s, t)$ for which $G$ is an undirected graph with maximum degree at most 2 containing a Hamilton path from node $s$ to node $t$.

(c) Given a universe $U$ and a collection of subsets $C = \{S_1, S_2, S_3, \ldots, S_n\}$, with each $S_i \subseteq U$, we say that a subset $H \subseteq U$ is a hitting set if each $S_i$ contains at least one element of $H$. In this problem we are interested in the case in which each $S_i$ has size at most 2. The language in question is

$$\text{HITTING SET-2} = \{(C, k) : \text{for all } S_i \in C, |S_i| \leq 2, \text{ and there is a hitting set } H \subseteq U \text{ with } |H| \leq k\}.$$ 

(d) The language consisting of 2-CNF formulas $\phi$ for which there exists an assignment that satisfies at least $3/4$ of the first 100 clauses, and all of the other clauses.

(e) The language consisting of 2-CNF formulas $\phi$ for which there exists an assignment that satisfies all of the first 100 clauses, and at least $3/4$ of the other clauses.