CS 153 Current topics in theoretical computer science	**Spring 2012**

Problem Set 2

Out: May 22 — *Due:* **May 29**

Reminder: you are encouraged to work in small groups; however you must turn in your own write-up and note with whom you worked. The solutions to some of these problems can be found in various online course notes and research papers. Please do not search for or refer to these solutions.

1. Show that the Hamilton Cycle problem HC defined by

$$\text{HC}_n(X_{1,1}, \cdots X_{n,n}) = \sum_{\sigma \in S_n \text{ is an } n\text{-cycle}} \prod_{i=1}^{n} X_{i,\sigma(i)}$$

is in VNP. Hint: Construct a polynomial in the entries of a matrix M that evaluates to 1 if M is a permutation matrix and 0 if it is any other 0/1 matrix. Find a condition for M being an n-cycle in terms of powers of M.

2. Viewing permanent

$$\text{PERM}_n = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)}$$

and determinant

$$\text{DET}_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} X_{i,\sigma(i)}$$

as non-commutative polynomials, prove that non-commutative arithmetic formulas for these two families require exponential size.

3. The *Pfaffian* of a $2n \times 2n$ matrix A is given by

$$\text{pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)}.$$

It is non-zero only on *skew symmetric* matrices A, in which case $\text{pf}(A)^2 = \det(A)$. Given an undirected planar graph $G = (V, E)$ on n nodes (n even), one can efficiently compute a skew-symmetric adjacency matrix A_G with entries in $\{0, X_{i,j}, -X_{i,j}\}$ for which

$$\text{pf}(A_G) = \sum_{M \subseteq E \text{ a perfect matching}} \prod_{(i,j) \in M} X_{i,j}.$$

Note that this is a *monotone* multilinear, homogeneous polynomial in the $X_{i,j}$ variables. Given a family of planar graphs G_n, it follows[1] from the fact that $\text{pf}(A_G) = \sqrt{\det(A_G)}$, and $\det(A_G)$ is in VP, that the family of polynomials $\{f_n(X) = \text{pf}(A_{G_n})\}$ is in VP.

[1] One needs to use a Taylor expansion trick to compute the square-root.

(a) Prove an exponential lower bound on the size of *monotone* circuits computing $\{f_n\}$. You will want to use the following lemmas which refer to $G_n = (V_n, E_n)$, the family of triangular grid graphs with sidelength n.

Lemma 2.1 *For every $t > 0$, there is a constant $\epsilon > 0$ such that the following holds: given a partition $V_n = S \cup T$ satisfying $|S| \leq |T| \leq 2|S|$, there exists a subset $E' \subseteq E_n$ with (i) $E' \geq \epsilon n$, (ii) this distance between every pair of edges in E' is at least t, and (iii) the distance of every edge in E' from the outer face of G is at least t.*

In other words, for every balanced cut in G_n, one can find a linear number of well-separated edges crossing the cut.

Lemma 2.2 *There is a distinguished vertex v in the interior of G_{22} with the following property for each edge e incident to v: if after deleting a subset of the vertices on the outer face of G_{22}, the resulting graph has at least one perfect matching excluding e, then the ratio of total perfect matchings to those excluding e is greater than 1.*

(b) Prove that there are polynomial size $(+, -, \times)$-circuits computing $\{f_n\}$ using only a *single* negation gate. Hint: reduce the number of negation gates to one, by induction on the size of the circuit.