CS151
Complexity Theory

Lecture 8

April 26, 2017

BPP

- How powerful is **BPP**?
- We have seen an example of a problem in

BPP

that we only know how to solve in **EXP**.

Is randomness a panacea for intractability?

BPP

- It is not known if **BPP = EXP** (or even **NEXP!**)
  - but there are strong hints that it does not
- Is there a deterministic simulation of **BPP** that does better than brute-force search?
  - yes, if allow non-uniformity

Theorem (Adleman): **BPP ⊂ P/poly**

BPP and Boolean circuits

- Proof:
  - language L ∈ **BPP**
  - error reduction gives TM M such that
    - if x ∈ L of length n

$$\Pr_y[M(x, y) \text{ accepts}] \geq 1 - (1/2)^{n^2}$$

    - if x ∉ L of length n

$$\Pr_y[M(x, y) \text{ rejects}] \geq 1 - (1/2)^{n^2}$$

BPP and Boolean circuits

– say "y is bad for x" if M(x,y) gives incorrect answer

– for fixed x: $\Pr_y[y \text{ is bad for } x] \leq (½)^{n^2}$

– $\Pr_y[y \text{ is bad for some } x] \leq 2^n(½)^{n^2} < 1$

– Conclude: there exists some y on which M(x, y) is always correct

– build circuit for M, hardwire this y

BPP and Boolean circuits

- Does **BPP** = **EXP** ?
- Adleman's Theorem shows:

BPP = **EXP** implies **EXP** $\subset$ **P/poly**

If you believe that randomness is all-powerful, you must also believe that non-uniformity gives an exponential advantage.

BPP

- Next:

  further explore the relationship between

  randomness

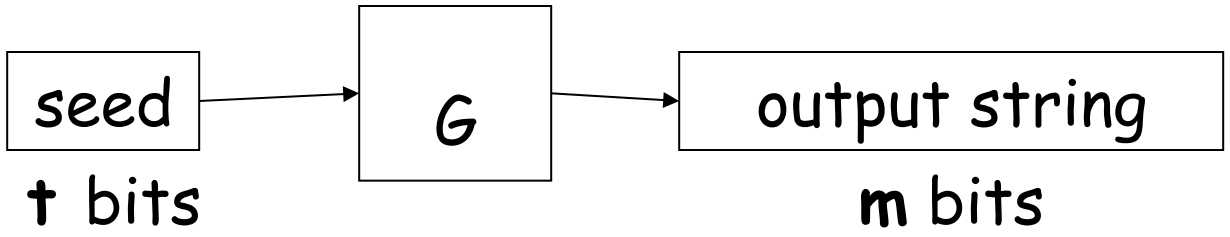
  and

  nonuniformity

- Main tool: pseudo-random generators

Derandomization

- Goal: try to simulate BPP in subexponential time (or better)
- use Pseudo-Random Generator (PRG):



- often: PRG "good" if it passes (ad-hoc) statistical tests

Derandomization

- ad-hoc tests not good enough to prove BPP has non-trivial simulations
- Our requirements:
  - G is efficiently computable
  - "stretches" **t** bits into **m** bits
  - "fools" small circuits: for all circuits C of size at most **s**:

$$|Pr_y[C(y) = 1] - Pr_z[C(G(z)) = 1]| \leq \boldsymbol{\varepsilon}$$

Simulating **BPP** using PRGs

- Recall: $L \in$ **BPP** implies exists p.p.t.TM M

  $x \in L \Rightarrow Pr_y[M(x,y) \text{ accepts}] \geq 2/3$

  $x \notin L \Rightarrow Pr_y[M(x,y) \text{ rejects}] \geq 2/3$

- given an input x:
  - convert M into circuit C(x, y)
  - simplification: pad y so that |C| = |y| = m
- hardwire input x to get circuit C_x

  $Pr_y[C_x(y) = 1] \geq 2/3$ ("yes")

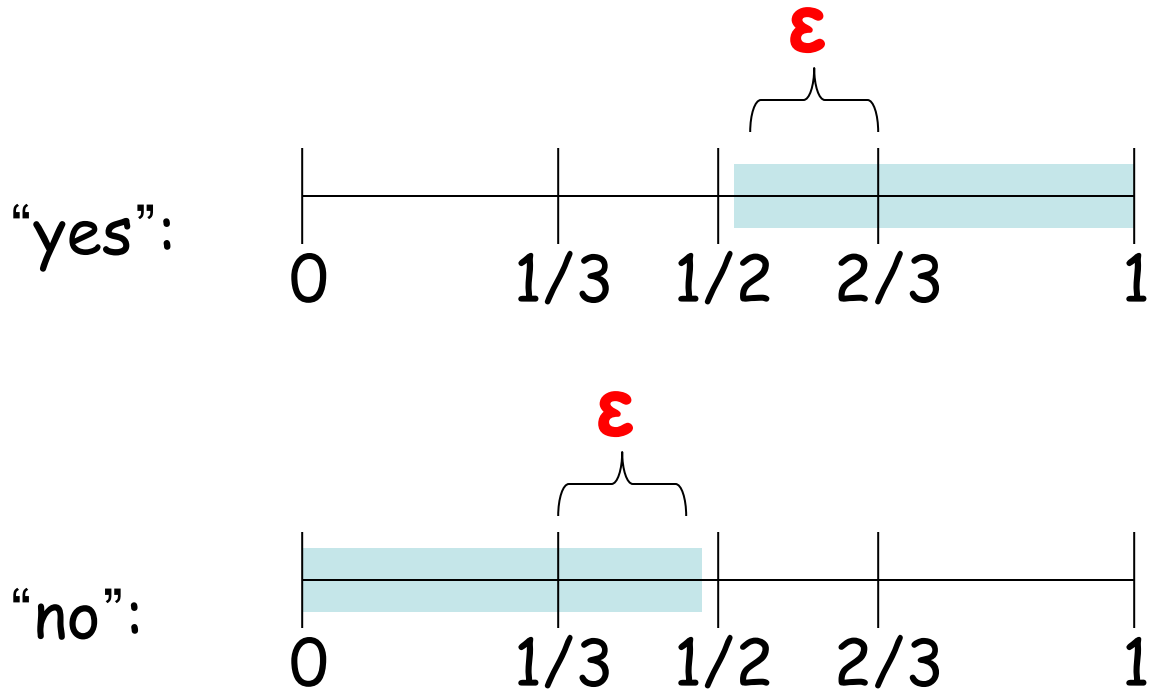
  $Pr_y[C_x(y) = 1] \leq 1/3$ ("no")

Simulating **BPP** using PRGs

- Use a PRG G with
  - output length **m**
  - seed length **t** « m
  - error **ε** < 1/6
  - fooling size **s** = m
- Compute $Pr_z[C_x(G(z)) = 1]$ exactly
  - evaluate $C_x(G(z))$ on every seed $z \in \{0,1\}^t$
- running time $(O(m)+(\text{time for } G))\mathbf{2^t}$

Simulating **BPP** using PRGs

- knowing $Pr_z[C_x(G(z)) = 1]$, can distinguish between two cases:



Blum-Micali-Yao PRG

- Initial goal: for all $1 > \delta > 0$, we will build a family of PRGs $\{G_m\}$ with:

  output length **m** fooling size **s** = m

  seed length **t =** m^{δ} running time m^{c}

  error **ε** < 1/6

- implies: **BPP** $\subset \bigcap_{\delta>0}$ **TIME(**$2^{n^{\delta}}$**)** $\subsetneq$ **EXP**
- Why? simulation runs in time

$$O(m+m^{c})(2^{m^{\delta}}) = O(2^{m^{2\delta}}) = O(2^{n^{2k\delta}})$$

Blum-Micali-Yao PRG

- PRGs of this type imply existence of one-way-functions
  - we'll use widely believed cryptographic assumptions

Definition: One Way Function (OWF): function family $f = \{f_n\}$, $f_n:\{0,1\}^n \rightarrow \{0,1\}^n$
  - f_n computable in poly(n) time
  - for every family of poly-size circuits $\{C_n\}$

$$Pr_x[C_n(f_n(x)) \in f_n^{-1}(f_n(x))] \leq \varepsilon(n)$$

  - $\varepsilon(n) = o(n^{-c})$ for all c

Blum-Micali-Yao PRG

- believe one-way functions exist
  - e.g. integer multiplication, discrete log, RSA (w/ minor modifications)

Definition: One Way Permutation: OWF in which f_n is 1-1

- can simplify "$Pr_x[C_n(f_n(x)) \in f_n^{-1}(f_n(x))] \leq \varepsilon(n)$" to

$$Pr_y[C_n(y) = f_n^{-1}(y)] \leq \varepsilon(n)$$

First attempt

- attempt at PRG from OWP f:
  - $t = m^{\delta}$
  - $y_0 \in \{0,1\}^t$
  - $y_i = f_t(y_{i-1})$
  - $G(y_0) = y_{k-1}y_{k-2}y_{k-3}\cdots y_0$
  - $k = m/t$
- computable in time at most

$$kt^c < mt^{c-1} = m^c$$

First attempt

- output is "**unpredictable**":
  - no poly-size circuit C can output y_{i-1} given $y_{k-1}y_{k-2}y_{k-3}\ldots y_i$ with non-negl. success prob.
  - if C could, then given y_i can compute $y_{k-1}, y_{k-2}, \ldots, y_{i+2}, y_{i+1}$ and feed to C
  - result is poly-size circuit to compute

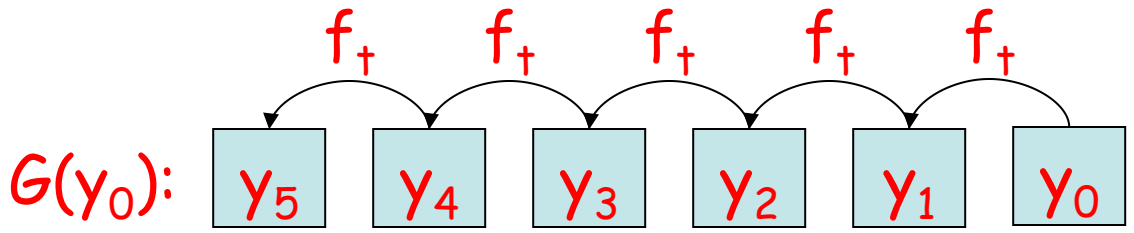
$$y_{i-1} = f_t^{-1}(y_i) \text{ from } y_i$$

  - note: we're using that f_t is 1-1

First attempt

attempt:

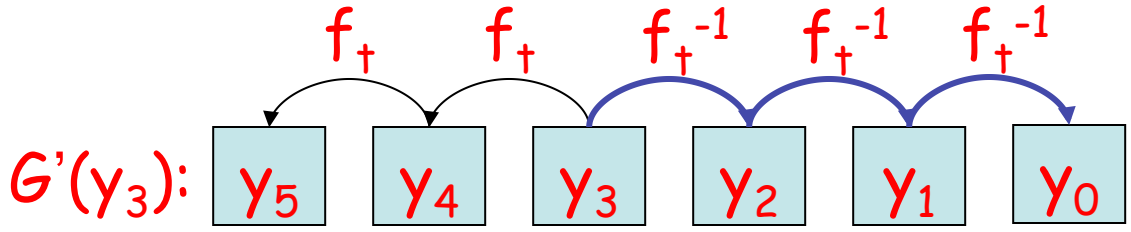
- $y_0 \in \{0,1\}^t$
- $y_i = f_t(y_{i-1})$



- $G(y_0) =$

  $y_{k-1}y_{k-2}y_{k-3}\cdots y_0$

same distribution!



First attempt

- one problem:
  - hard to compute y_{i-1} from y_i
  - but might be easy to compute single bit (or several bits) of y_{i-1} from y_i
  - could use to build small circuit C that distinguishes G's output from uniform distribution on $\{0,1\}^m$

First attempt

- second problem
  - we don't know if "unpredictability" given a prefix is sufficient to meet fooling requirement:

$$|Pr_y[C(y) = 1] - Pr_z[C(G(z)) = 1]| \leq \boldsymbol{\varepsilon}$$

Hard bits

- If $\{f_n\}$ is one-way permutation we know:
  - no poly-size circuit can compute $f_n^{-1}(y)$ from y with non-negligible success probability

$$Pr_y[C_n(y) = f_n^{-1}(y)] \leq \varepsilon'(n)$$

- We want to identify a single bit position j for which:
  - no poly-size circuit can compute $(f_n^{-1}(x))_j$ from x with non-negligible advantage over a coin flip

$$Pr_y[C_n(y) = (f_n^{-1}(y))_j] \leq \frac{1}{2} + \varepsilon(n)$$

Hard bits

- For some specific functions f we know of such a bit position j

- More general:

  function $h_n:\{0,1\}^n \rightarrow \{0,1\}$

  rather than just a bit position j.

Hard bits

Definition: hard bit for $g = \{g_n\}$ is family $h = \{h_n\}$, $h_n:\{0,1\}^n \rightarrow \{0,1\}$ such that if circuit family $\{C_n\}$ of size $s(n)$ achieves:

$$Pr_x[C_n(x) = h_n(g_n(x))] \geq ½ + \varepsilon(n)$$

then there is a circuit family $\{C'_n\}$ of size $s'(n)$ that achieves:

$$Pr_x[C'_n(x) = g_n(x)] \geq \varepsilon'(n)$$

with:

- $\varepsilon'(n) = (\varepsilon(n)/n)^{O(1)}$
- $s'(n) = (s(n)n/\varepsilon(n))^{O(1)}$

Goldreich-Levin

- To get a generic hard bit, first need to modify our one-way permutation

- Define $f'_n : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^{2n}$ as:

$$f'_n(x,y) = (f_n(x), y)$$

Goldreich-Levin

$f'_n(x,y) = (f_n(x), y)$

- Two observations:
  - f' is a permutation if f is
  - if circuit C_n achieves

$$Pr_{x,y}[C_n(x,y) = f'^{-1}_n(x,y)] \geq \varepsilon(n)$$

then for some y^*

$$Pr_x[C_n(x,y^*)=f'^{-1}_n(x,y^*)=(f_n^{-1}(x), y^*)] \geq \varepsilon(n)$$

and so f' is a one-way permutation if f is.

Goldreich-Levin

- The Goldreich-Levin function:

$$GL_{2n} : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$$

  is defined by:

$$GL_{2n}(x,y) = \oplus_{i:y_i = 1} x_i$$

  – parity of subset of bits of x selected by 1's of y
  – inner-product of n-vectors x and y in GF(2)

Theorem (G-L): for every function f, GL is a hard bit for f'. (proof: problem set)

Distinguishers and predictors

- Distribution D on $\{0,1\}^n$
- D **ε**-passes statistical tests of size s if for all circuits of size s:

$$|Pr_{y \leftarrow U_n}[C(y) = 1] - Pr_{y \leftarrow D}[C(y) = 1]| \leq \varepsilon$$

  – circuit violating this is sometimes called an efficient "distinguisher"

Distinguishers and predictors

- D **ε**-passes prediction tests of size s if for all circuits of size s:

$$\Pr_{y \leftarrow D}[C(y_{1,2,\ldots,i-1}) = y_i] \leq \tfrac{1}{2} + \varepsilon$$

  – circuit violating this is sometimes called an efficient "predictor"
- predictor seems stronger
- Yao showed essentially the same!
  – important result and proof ("hybrid argument")

Distinguishers and predictors

Theorem (Yao): if a distribution D on $\{0,1\}^n$ (ε/n)-passes all prediction tests of size s, then it ε-passes all statistical tests of size $s' = s - O(n)$.

Distinguishers and predictors

- Proof:
  - idea: proof by contradiction
  - given a size s' distinguisher C:

$$|Pr_{y \leftarrow U_n}[C(y) = 1] - Pr_{y \leftarrow D}[C(y) = 1]| > \varepsilon$$

  - produce size s predictor P:

$$Pr_{y \leftarrow D}[P(y_{1,2,\ldots,i-1}) = y_i] > \tfrac{1}{2} + \varepsilon/n$$

  - work with distributions that are "hybrids" of the uniform distribution U_n and D

Distinguishers and predictors

– given a size s' distinguisher C:

$$|Pr_{y \leftarrow U_n}[C(y) = 1] - Pr_{y \leftarrow D}[C(y) = 1]| > \varepsilon$$

– define n+1 hybrid distributions

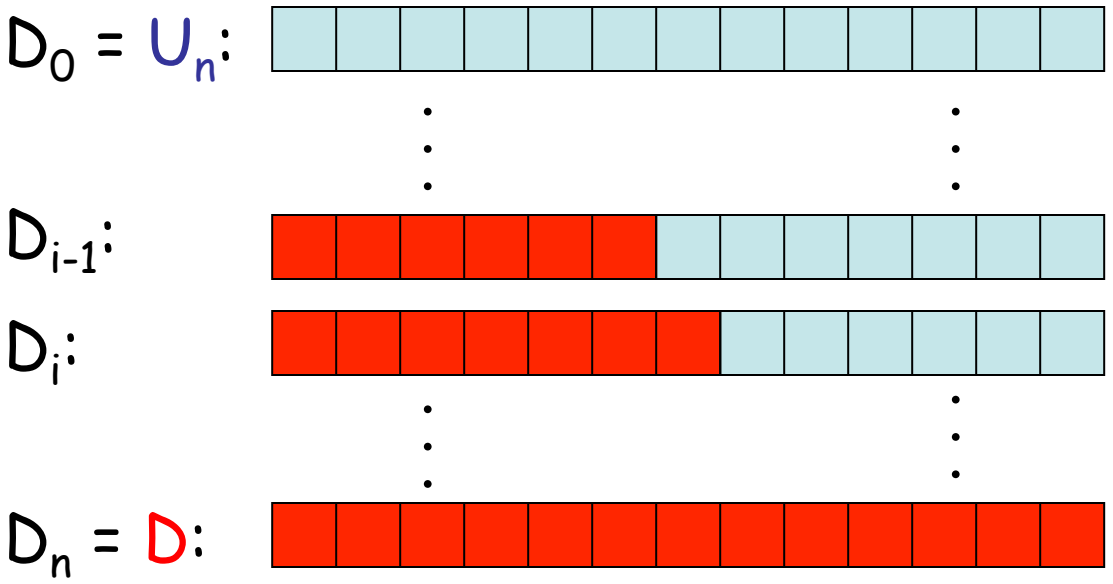
– hybrid distribution D_i:

- sample $b = b_1 b_2 \ldots b_n$ from D
- sample $r = r_1 r_2 \ldots r_n$ from U_n
- output:

$$b_1 b_2 \ldots b_i \, r_{i+1} r_{i+2} \ldots r_n$$

Distinguishers and predictors

- Hybrid distributions:



Distinguishers and predictors

– Define: $p_i = Pr_{y \leftarrow D_i}[C(y) = 1]$

– Note: $p_0 = Pr_{y \leftarrow U_n}[C(y)=1]$; $p_n = Pr_{y \leftarrow D}[C(y)=1]$

– by assumption: $\varepsilon < |p_n - p_0|$

– triangle inequality: $|p_n - p_0| \leq \Sigma_{1 \leq i \leq n} |p_i - p_{i-1}|$

– there must be some i for which

$$|p_i - p_{i-1}| > \varepsilon/n$$

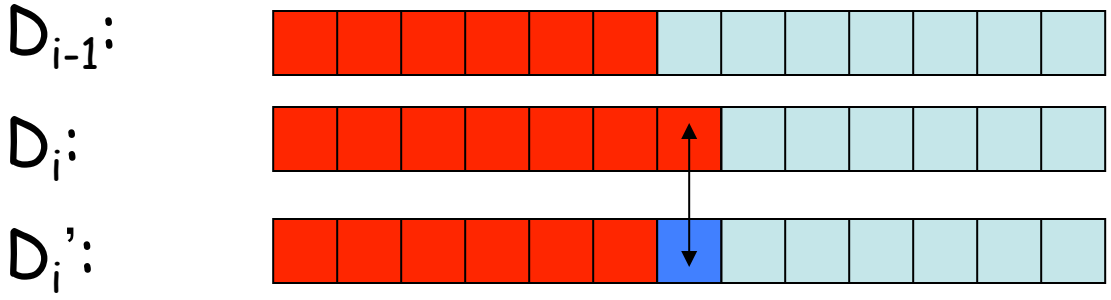
– WLOG assume $p_i - p_{i-1} > \varepsilon/n$

- can invert output of C if necessary

Distinguishers and predictors

– define distribution D_i' to be D_i with i-th bit flipped

– $p_i' = Pr_{y \leftarrow D_i'}[C(y) = 1]$



– notice:

$$D_{i-1} = (D_i + D_i')/2 \qquad p_{i-1} = (p_i + p_i')/2$$

Distinguishers and predictors

- randomized predictor P' for i^{th} bit:
  - input: $u = y_1 y_2 \ldots y_{i-1}$ (which comes from D)
  - flip a coin: $\mathbf{d} \in \{0,1\}$
  - $w = w_{i+1} w_{i+2} \ldots w_n \leftarrow U_{n-i}$
  - evaluate $C(u\mathbf{d}w)$
  - if 1, output d; if 0, output $\neg d$

Claim:

$$Pr_{y \leftarrow D, d, w \leftarrow U_{n-i}}[P'(y_1 \ldots_{i-1}) = y_i] > \frac{1}{2} + \varepsilon/n.$$

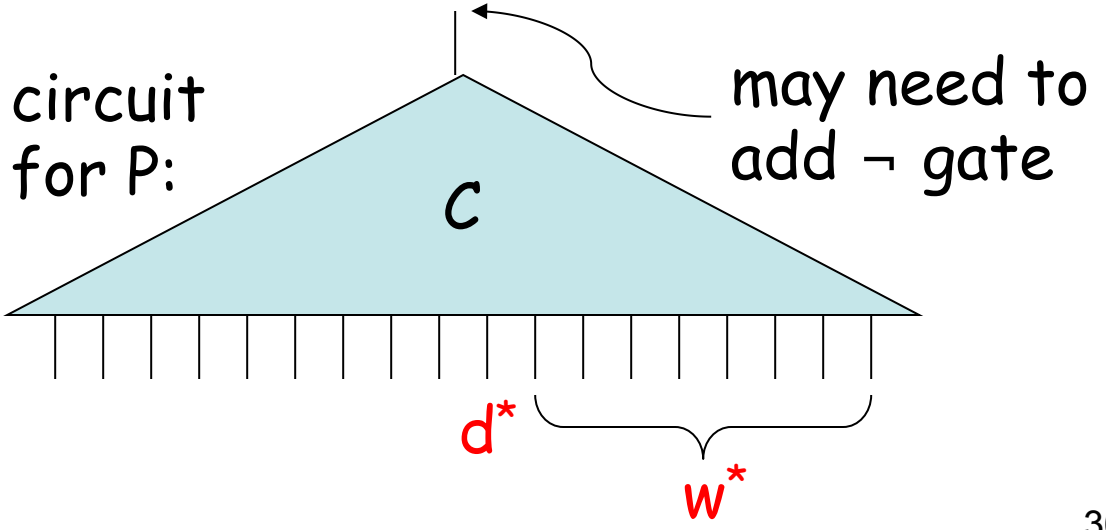
Distinguishers and predictors

- P' is randomized procedure
- there must be some fixing of its random bits d, w that preserves the success prob.
- final predictor P has d^* and w^* hardwired:

Size is

$s' + O(n) = s$

as promised



Distinguishers and predictors

$u = y_1 y_2 \cdots y_{i-1}$

- Proof of claim:

$$\Pr_{y \leftarrow D, d, w \leftarrow U_{n-i}}[P'(y_1 \cdots_{i-1}) = y_i] =$$

$$\Pr[y_i = d \mid C(u,d,w) = 1]\Pr[C(u,d,w) = 1]$$

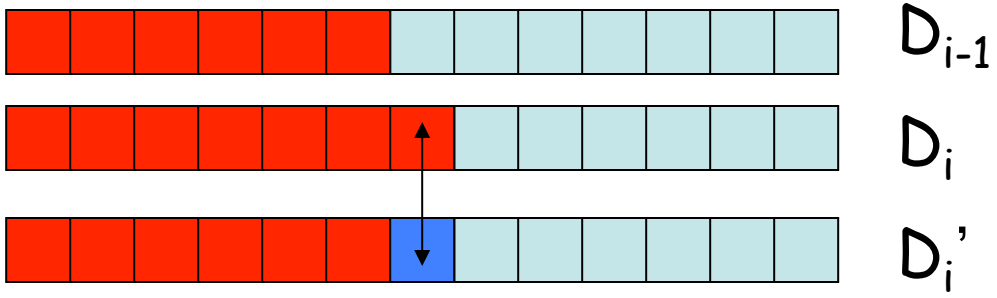
$$+ \Pr[y_i = \neg d \mid C(u,d,w) = 0]\Pr[C(u,d,w) = 0]$$

$$= \Pr[y_i = d \mid C(u,d,w) = 1](p_{i-1})$$

$$+ \Pr[y_i = \neg d \mid C(u,d,w) = 0](1 - p_{i-1})$$

Distinguishers and predictors

$u = y_1 y_2 \ldots y_{i-1}$



– Observe:

$\Pr[y_i = d \mid C(u,d,w) = 1]$

$= \Pr[C(u,d,w) = 1 \mid y_i = d]\Pr[y_i=d] / \Pr[C(u,d,w) = 1]$

$= p_i/(2p_{i-1})$

$\Pr[y_i = \neg d \mid C(u,d,w) = 0]$

$= \Pr[C(u,d,w) = 0 \mid y_i = \neg d]\Pr[y_i=\neg d] / \Pr[C(u,d,w) = 0]$

$= (1 - p_i') / 2(1 - p_{i-1})$

Distinguishers and predictors

- Success probability:

  $\Pr[y_i=d|C(u,d,w)=1](p_{i-1}) + \Pr[y_i=\neg d|C(u,d,w)=0](1-p_{i-1})$

- We know:
  - $\Pr[y_i = d \mid C(u,d,w) = 1] = p_i/(2p_{i-1})$
  - $\Pr[y_i = \neg d \mid C(u,d,w) = 0] = (1 - p_i')/2(1 - p_{i-1})$
  - $p_{i-1} = (p_i + p_i')/2$
  - $p_i - p_{i-1} > \varepsilon/n$

$p_i'/2 = p_{i-1} - p_i/2$

- Conclude:

$$\Pr[P'(y_{1\ldots i-1}) = y_i] = ½ + (p_i - p_i')/2$$

$$= ½ + p_i/2 - (p_{i-1} - p_i/2) = ½ + p_i - p_{i-1} > ½ + \varepsilon/n.$$

The BMY Generator

- Recall goal: for all $1 > \delta > 0$, family of PRGs $\{G_m\}$ with

  output length **m** fooling size **s** = m

  seed length **t =** m^{δ} running time m^{c}

  error **ε** < 1/6

- If one way permutations exist then WLOG there is OWP $f = \{f_n\}$ with hard bit $h = \{h_n\}$

The BMY Generator

- Generator $G^{\delta} = \{G^{\delta}_m\}$:
  - $t = m^{\delta}$
  - $y_0 \in \{0,1\}^t$
  - $y_i = f_t(y_{i-1})$
  - $b_i = h_t(y_i)$
  - $G^{\delta}(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

The BMY Generator

Theorem (BMY): for every $\delta > 0$, there is a constant c s.t. for all d, e, G^{δ} is a PRG with

error $\boldsymbol{\varepsilon} < 1/m^d$

fooling size $\mathbf{s} = m^e$

running time m^c

- Note: stronger than we needed
  - sufficient to have $\boldsymbol{\varepsilon} < 1/6$; $\mathbf{s} = m$

The BMY Generator

Generator $G^\delta = \{G^\delta_m\}$:

- $t = m^\delta$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^\delta_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

- Proof:
  - computable in time at most

$$mt^c < m^{c+1}$$

  - assume G^δ does not $(1/m^d)$-pass statistical test $C = \{C_m\}$ of size m^e:

$$|Pr_{y \leftarrow U_m}[C(y) = 1] - Pr_{z \leftarrow D}[C(z) = 1]| > 1/m^d$$

The BMY Generator

Generator $G^\delta = \{G^\delta_m\}$:

- $t = m^\delta$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^\delta_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

– transform this **distinguisher** into a **predictor** P of size $m^e + O(m)$:

$$\Pr_y[P(b_{m-1}\ldots b_{m-i}) = b_{m-i-1}] > \tfrac{1}{2} + 1/m^{d+1}$$

The BMY Generator

Generator $G^{\delta} = \{G^{\delta}_m\}$:

- $t = m^{\delta}$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^{\delta}_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

– a procedure to compute $h_t(f_t^{-1}(y))$

- set $y_{m-i} = y$; $b_{m-i} = h_t(y_{m-i})$
- compute y_j, b_j for j = m-i+1, m-i+2…, m-1 as above
- evaluate $P(b_{m-1}b_{m-2}\ldots b_{m-i})$
- f a permutation implies $b_{m-1}b_{m-2}\ldots b_{m-i}$ distributed as (prefix of) output of generator:

$$Pr_y[P(b_{m-1}b_{m-2}\ldots b_{m-i}) = b_{m-i-1}] > \tfrac{1}{2} + 1/m^{d+1}$$

The BMY Generator

Generator $G^\delta = \{G^\delta_m\}$:

- $t = m^\delta$; $y_0 \in \{0,1\}^t$; $y_i = f_t(y_{i-1})$; $b_i = h_t(y_i)$
- $G^\delta_m(y_0) = b_{m-1}b_{m-2}b_{m-3}\ldots b_0$

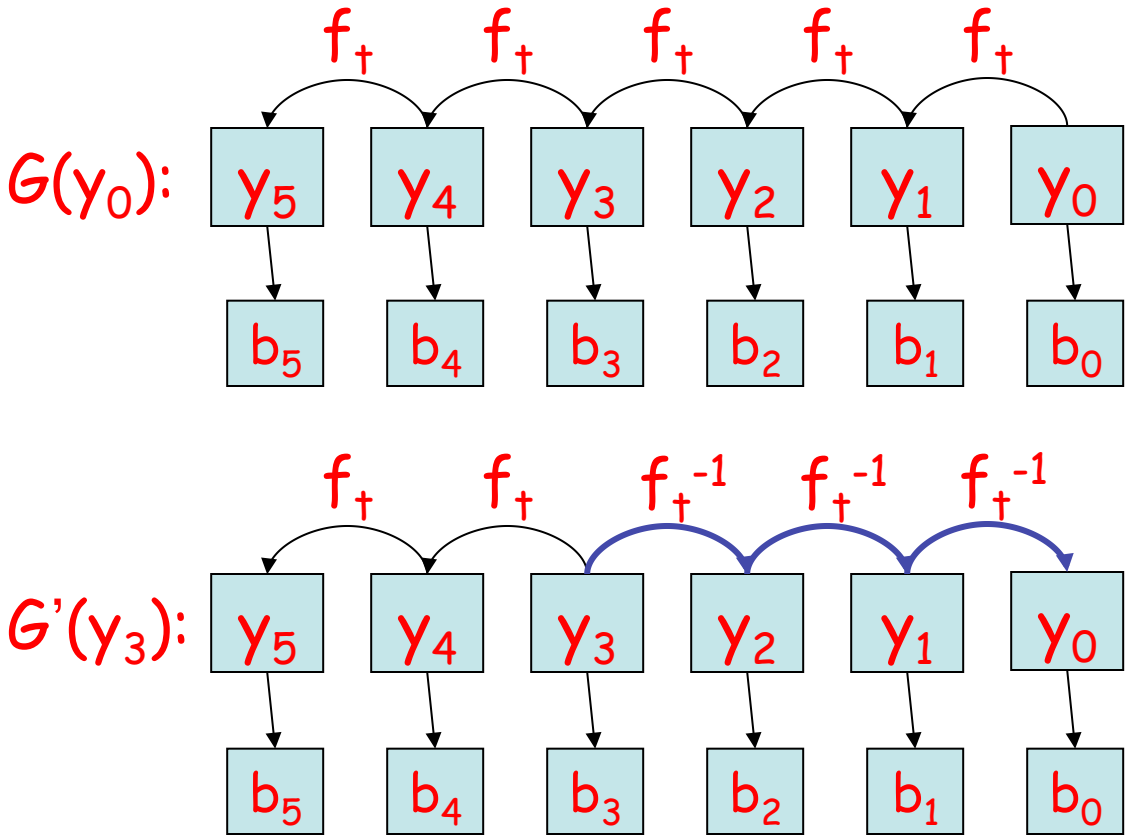
$$\Pr_y[P(b_{m-1}b_{m-2}\ldots b_{m-i}) = b_{m-i-1}] > \tfrac{1}{2} + 1/m^{d+1}$$

- What is b_{m-i-1}?

$$b_{m-i-1} = h_t(y_{m-i-1}) = h_t(f_t^{-1}(y_{m-i})) = h_t(f_t^{-1}(y))$$

- We have described a family of polynomial-size circuits that computes $h_t(f_t^{-1}(y))$ from y with success greater than $\frac{1}{2} + 1/\text{poly}(m)$
- Contradiction.

The BMY Generator



same distribution

Hardness vs. randomness

- We have shown:

  If one-way permutations exist then

  $$\mathbf{BPP} \subset \bigcap_{\delta>0} \mathbf{TIME}(2^{n^{\delta}}) \subsetneq \mathbf{EXP}$$

- simulation is better than brute force, but just barely
- stronger assumptions on difficulty of inverting OWF lead to better simulations…