CS151
Complexity Theory

Lecture 7

April 20, 2021

Clique

CLIQUE = { (G, k) | G is a graph with a clique of size ≥ k }

(clique = set of vertices every pair of which are connected by an edge)

- CLIQUE is **NP**-complete.

Circuit lower bounds

- We think that **NP** requires exponential-size circuits.
- Where should we look for a problem to attempt to prove this?

- Intuition: "hardest problems" – i.e., **NP-**complete problems

Circuit lower bounds

- Formally:
  - if *any* problem in **NP** requires super-polynomial size circuits
  - then *every* **NP**-complete problem requires super-polynomial size circuits

  - Proof idea: poly-time reductions can be performed by poly-size circuits using a variant of CVAL construction

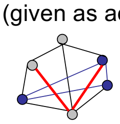
Monotone problems

- Definition: monotone language = language $L \subseteq \{0,1\}^*$ such that $x \in L$ implies $x' \in L$ for all $x \preceq x'$.

  - flipping a bit of the input from 0 to 1 can only change the output from "no" to "yes" (or not at all)

Monotone problems

- some **NP**-complete languages are monotone
  - e.g. CLIQUE (given as adjacency matrix):



  - others: HAMILTON CYCLE, SET COVER…
  - but not SAT, KNAPSACK…

Monotone circuits

A restricted class of circuits:

- Definition: monotone circuit = circuit whose gates are ANDs ($\wedge$), ORs ($\vee$), but no NOTs
- can compute exactly the monotone fns.
  - monotone functions closed under AND, OR

Monotone circuits

- A question:

  Do all poly-time computable monotone functions have poly-size monotone circuits?

  - recall: true in non-monotone case

Monotone circuits

A monotone circuit for $\text{CLIQUE}_{n,k}$

- Input: graph $G = (V,E)$ as adj. matrix, $|V|=n$
  - variable $x_{i,j}$ for each possible edge (i,j)
- ISCLIQUE(S) = monotone circuit that = 1 iff $S \subseteq V$ is a clique: $\bigwedge_{i,j \in S} x_{i,j}$

  $\text{CLIQUE}_{n,k}$ computed by monotone circuit:

  $$\bigvee_{S \subseteq V, |S|=k} \text{ISCLIQUE(S)}$$

Monotone circuits

- Size of this monotone circuit for $\text{CLIQUE}_{n,k}$: $\binom{n}{k}\binom{k}{2}$
- when $k = n^{1/4}$, size is approximately:

$$\left(\frac{n}{n^{1/4}}\right)^{n^{1/4}} \left(\frac{n^{1/4}}{2}\right)^{2} \approx n^{\Omega(n^{1/4})}$$

Monotone circuits

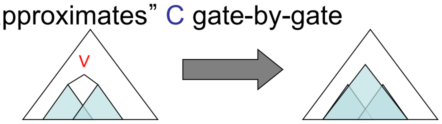
- Theorem (Razborov 85): monotone circuits for $\text{CLIQUE}_{n,k}$ with $k = n^{1/4}$ must have size at least

  $$2^{\Omega(n^{1/8})}.$$

- Proof:
  - rest of lecture

Proof idea

- "method of approximation"
- suppose C is a monotone circuit for $\text{CLIQUE}_{n,k}$
- build another monotone circuit CC that "approximates" C gate-by-gate



Proof idea

- on test collection of positive/negative instances of $CLIQUE_{n,k}$:
  - local property: few errors at each gate
  - global property: many errors on test collection
- Conclude: C has many gates

Notation

- input: graph G = (V, E)
- variable $x_{j,k}$ for each potential edge (j, k)
- $CC(X_1, X_2, \ldots X_m)$, where $X_i \subseteq V$, means:

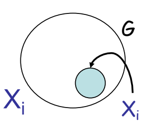
$$\bigvee_i(\bigwedge_{j,k \in X_i} x_{j,k}) \; *$$

- For example: $CC(X_1, X_2, \ldots X_m)$ where the X_i range over all k-subsets of V
  - this is the obvious monotone circuit for $CLIQUE_{n,k}$ from a previous slide.

*[CC( ) = 0; ($\bigwedge_{i,j \in \emptyset} x_{i,j}$)= 1]

Preview

- approximate circuit $CC(X_1, X_2, \ldots X_m)$
- n = # nodes
- k = $n^{1/4}$ = size of clique
- h = $n^{1/8}$ = max. size of subsets X_i



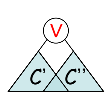
  - this is "global property" that ensures lots of errors
  - many graphs G with no k-cliques, but clique on X_i of size h

Preview

- approximate circuit $CC(X_1, X_2, \ldots X_m)$
- p = $n^{1/8}\log n$
- M = $(p - 1)^h h!$
- max # of subsets is M (so m ≤ M)
  - critical for "local property" that ensures few errors at each gate

Building CC

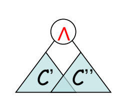
- CC ("crude circuit") for circuit C defined inductively as follows:
  - CC for single variable $x_{j,k}$ is just CC({ j, k })
    - no errors yet!
  - CC for circuit C of form:



  - "approximate OR" of CC for C', CC for C''

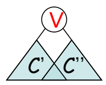
Building CC

- CC for circuit C of form:



- "approximate AND" of CC for C', CC for C''
- "approximate OR" and "approximate AND" steps introduce errors

Approximate OR



CC($X_1,X_2,\ldots X_{m'}$)   CC($Y_1,Y_2,\ldots Y_{m''}$)

- exact OR:

  CC($X_1,X_2,\ldots X_{m'},Y_1,Y_2,\ldots Y_{m''}$)

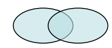
  – set sizes still ≤ h

  – may be up to 2M sets; need to reduce to M

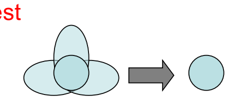
Approximate OR

– throw away sets? bad:many errors

– throw away overlapping sets? – better

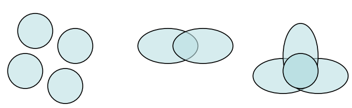


– throw away special configuration of overlapping sets – best



Sunflowers

- Definition: (h, p)-sunflower is a family of p sets, each of size at most h, such that intersection of every pair is a subset S (the “core”).

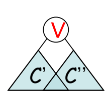


Sunflowers

Lemma (Erdös-Rado): Every family of more than $M = (p-1)^h h!$ sets, each of size at most h, contains an (h, p)-sunflower.

- Proof:
  – not hard
  – in Papadimitriou, elsewhere

Approximate OR



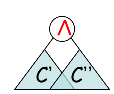
- CC($X_1,X_2,\ldots X_{m'}$)
- CC($Y_1,Y_2,\ldots Y_{m''}$)
- exact OR:

  CC($X_1,X_2,\ldots X_{m'},Y_1,Y_2,\ldots Y_{m''}$)

  – while more than M sets, find (h, p)-sunflower; replace with its core (“**pluck**”)
- approximate OR:

  CC(**pluck**($X_1,X_2,\ldots X_{m'},Y_1,Y_2,\ldots Y_{m''}$))

Approximate AND



- CC($X_1,X_2,\ldots X_{m'}$)
- CC($Y_1,Y_2,\ldots Y_{m''}$)
- (close to) exact AND:

  CC($\{(X_i \cup Y_j) : 1 \le i \le m', 1 \le j \le m''\}$)

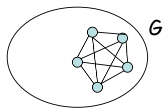
  – some sets may be larger than h; discard them

  – may be up to M^2 sets. While > M sets, find (h, p)-sunflower; replace with its core (“**pluck**”)
- approximate AND:

  CC(**pluck** ($\{(X_i \cup Y_j) : |X_i \cup Y_j| \le h\}$))

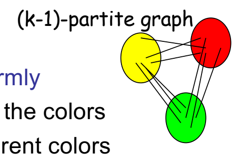
Test collection

- Positive instances: all graphs G on n nodes with a k-clique and no other edges.



Test collection

- Negative instances:
  - k-1 colors
  - color each node uniformly at random with one of the colors
  - edge (x, y) iff x, y different colors
  - no k-clique
  - include graphs in their multiplicities
    - makes analysis easier

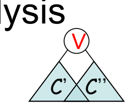


"Local" analysis

- "false positive":
  - negative example
  - gate is supposed to output 0, but our CC outputs 1

Lemma: each approximation step introduces at most $M^2(k-1)^n/2^p$ false positives.

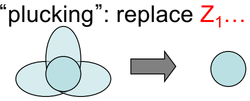
"Local" analysis



- Proof:
  - case 1: OR

    $CC(X_1,X_2,\ldots X_{m'})$ $CC(Y_1,Y_2,\ldots Y_{m''})$

    $CC(\textbf{pluck}(X_1,X_2,\ldots X_{m'},Y_1,Y_2,\ldots Y_{m''}))$
  - given "plucking": replace $Z_1\ldots Z_p$ with Z



  - bad case: clique on Z, and each petal is missing at least one edge

"Local" analysis

- what is the probability of a repeated color in each Z_i but no repeated colors in Z?

$\Pr[R(Z_1)\wedge R(Z_2)\ldots R(Z_p) \wedge \neg R(Z)]$

$\leq \Pr[R(Z_1)\wedge R(Z_2)\ldots R(Z_p) | \neg R(Z)]$

event R(S) = repeated colors in S

(definition of conditional probability)

$= \prod_i \Pr[R(Z_i) \mid \neg R(Z)]$

(independent events given no repeats in Z)

$\leq \prod_i \Pr[R(Z_i)]$

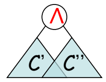
(obviously larger)

"Local" analysis

- for every pair of vertices in Z_i, probability of same color is $1/(k-1)$
- $R(Z_i) \leq$ (h choose 2)/(k-1) $\leq$ ½
- $\prod_i \Pr[R(Z_i)] \leq (½)^p$
- \# negative examples is $(k-1)^n$
- \# false positives in given plucking step is at most $(½)^p(k-1)^n$
- at most M plucking steps
- \# false positives at OR $\leq M(½)^p(k-1)^n$

“Local” analysis

– case 2: AND



$CC(X_1, X_2, \ldots X_{m'})$ $CC(Y_1, Y_2, \ldots Y_{m''})$

$CC($**pluck**$(\{(X_i \cup Y_j) : |X_i \cup Y_j| \leq h \}))$

– discarding sets $(X_i \cup Y_j)$ larger than h can only make circuit accept fewer examples

- no false positives here

“Local” analysis

– up to M^2 pluckings

– each introduces at most

$$(½)^p(k-1)^n$$

false positives (previous slides)

– # false positives at AND $\leq M^2(½)^p(k-1)^n$

“Local” analysis

- “false negative”:
  – positive example;
  – gate is supposed to output 1, but our CC outputs 0

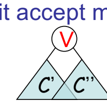
Lemma: each approximation step introduces at most

$$M^2 \binom{n-h-1}{k-h-1}$$

false negatives.

“Local” analysis

- Proof:
  – Case 1: OR
  – plucking can only make circuit accept more examples
    - no false negatives here.



  – Case 2: AND

$CC(X_1, X_2, \ldots X_{m'})$ $CC(Y_1, Y_2, \ldots Y_{m''})$

$CC($**pluck**$(\{(X_i \cup Y_j) : |X_i \cup Y_j| \leq h \}))$

- for positive examples: clique on X_i and clique on Y_j $\Rightarrow$ clique on $X_i \cup Y_j$ (no false negatives until discard $X_i \cup Y_j$ sets)

“Local” analysis

– discarding set $Z = (X_i \cup Y_j)$ larger than h may introduce false negatives

– any clique that includes Z is a problem; there are at most

$$\binom{n-|Z|}{k-|Z|} \leq \binom{n-h-1}{k-h-1}$$

such positive examples, since $|Z|>h$ & $h<<k$

– at most M^2 such deletions

– we’ve seen plucking doesn’t matter

“Global” analysis

Lemma: every non-trivial CC outputs 1 on at least ½ of the negative examples.

- Proof:
  – CC contains some set X of size at most h
  – accepts all neg. examples with different colors in X
  – probability X has repeated colors is

$$R(X) \leq (h \text{ choose } 2)/(k-1) \leq ½$$

  – probability over negative examples that CC accepts is at least ½.

Finishing up

- First possibility: trivial CC, rejects all positive examples
  - every positive example must have been false negative at some gate
  - number of gates must be at least:

# of positive examples

$$\binom{n}{k} / M^2 \binom{n-h-1}{k-h-1}$$

false negatives at each gate

Finishing up

- Second possibility: CC accepts at least ½ of negative examples
  - every negative example must have been false positive at some gate
  - number of gates must be at least:

$$\frac{1}{2}(k-1)^n / M^2 2^{-p}(k-1)^n$$

# of negative examples

false positives at each gate

Finishing up

$$\binom{n}{k} / M^2 \binom{n-h-1}{k-h-1}$$

$$\frac{1}{2}(k-1)^n / M^2 2^{-p}(k-1)^n$$

Both quantities are at least $2^{\Omega(n^{1/8})}$

Conclusions

- A question (true in non-monotone case):

  Do all
  poly-time computable monotone functions
  have
  poly-size monotone circuits?

- if yes, then we would have just proved **P ≠ NP**
  - why?

Conclusions

- unfortunately, answer is no
- Razborov later showed similar (super-polynomial) lower bound for MATCHING, which is in **P**…