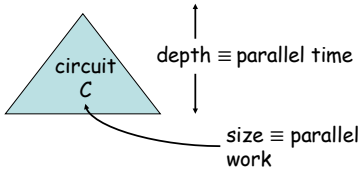
CS151
Complexity Theory

Lecture 6

April 15, 2021

Parallelism

- uniform circuits allow refinement of polynomial time:



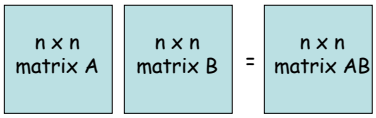
Parallelism

- the **NC** ("**N**ick's **C**lass") Hierarchy (of logspace uniform circuits):

  $\mathbf{NC_k}$ = O($\log^k$ n) depth, poly(n) size

  $\mathbf{NC} = \cup_k \mathbf{NC_k}$
- captures "efficiently parallelizable problems"
- not realistic? overly generous
- OK for proving non-parallelizable

Matrix Multiplication



- what is the parallel complexity of this problem?
  - work = poly(n)
  - time = $\log^k$(n)? (which k?)

Matrix Multiplication

- two details
  - arithmetic matrix multiplication…

    A = $(a_{i,k})$ B = $(b_{k,j})$ $(AB)_{i,j} = \Sigma_k (a_{i,k} \times b_{k,j})$

    … vs. Boolean matrix multiplication:

    A = $(a_{i,k})$ B = $(b_{k,j})$ $(AB)_{i,j} = \vee_k (a_{i,k} \wedge b_{k,j})$

  - single output bit: to make matrix multiplication a language: on input A, B, (i, j) output $(AB)_{i,j}$

Matrix Multiplication

- Boolean Matrix Multiplication is in $\mathbf{NC_1}$
  - level 1: compute n ANDS: $a_{i,k} \wedge b_{k,j}$
  - next log n levels: tree of ORS
  - n^2 subtrees for all pairs (i, j)
  - select correct one and output

Boolean formulas and $\mathbf{NC_1}$

- Previous circuit is actually a formula. This is no accident:

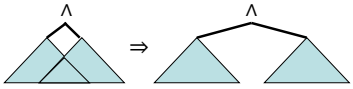
Theorem: L ∈ $\mathbf{NC_1}$ iff decidable by polynomial-size uniform* family of Boolean formulas.

* DSPACE($\log^2$ n)-uniform

Note: we measure formula size by leaf-size.

Boolean formulas and $\mathbf{NC_1}$

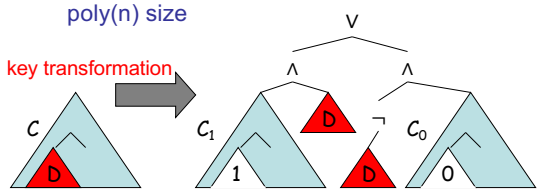
- Proof:
  - (⇒) convert $\mathbf{NC_1}$ circuit into formula
    - recursively:



    - note: logspace transformation (stack depth log n, stack record 1 bit – "left" or "right")

Boolean formulas and $\mathbf{NC_1}$

- (⇐) convert formula of size n into formula of depth O(log n)
  - note: size ≤ 2^{depth}, so new formula has poly(n) size



Boolean formulas and $\mathbf{NC_1}$

- D any minimal subtree with size at least n/3
  - implies size(D) ≤ 2n/3
- define T(n) = maximum depth required for any size n formula
- C_1, C_0, D all size ≤ 2n/3

$$T(n) \le T(2n/3) + 3$$

$$\text{implies } T(n) \le O(\log n)$$

Relation to other classes

- Clearly $\mathbf{NC} \subseteq \mathbf{P}$
  - recall $\mathbf{P}$ ≡ uniform poly-size circuits
- $\mathbf{NC_1} \subseteq \mathbf{L}$
  - on input x, compose logspace algorithms for:
    - generating $C_{|x|}$
    - converting to formula
    - FVAL

Relation to other classes

- $\mathbf{NL} \subseteq \mathbf{NC_2}$: S-T-CONN ∈ $\mathbf{NC_2}$
  - given G = (V, E), vertices s, t
  - A = adjacency matrix (with self-loops)
  - $(A^2)_{i,j}$ = 1 iff path of length ≤ 2 from node i to node j
  - $(A^n)_{i,j}$ = 1 iff path of length ≤ n from node i to node j
  - compute with depth log n tree of Boolean matrix multiplications, output entry s, t
  - $\log^2$ n depth total

NC vs. P

- can every efficient algorithm be efficiently parallelized?

$$\mathbf{NC} \stackrel{?}{=} \mathbf{P}$$

- **P**-complete problems least-likely to be parallelizable
  - if **P**-complete problem is in **NC**, then **P** = **NC**
  - Why?

    we use logspace reductions to show problem **P**-complete; **L** in **NC**

NC vs. P

- can every uniform, poly-size Boolean circuit family be converted into a uniform, poly-size Boolean formula family?

$$\mathbf{NC_1} \stackrel{?}{=} \mathbf{P}$$

NC Hierarchy Collapse

$$NC_1 \subseteq NC_2 \subseteq NC_3 \subseteq NC_4 \subseteq \ldots \subseteq NC$$

Exercise

if $NC_i = NC_{i+1}$, then $NC = NC_i$

(prove for non-uniform versions of classes)

Lower bounds

- Recall: "**NP** does not have polynomial-size circuits" (**NP** $\subsetneq$ **P/poly**) implies **P** $\neq$ **NP**
- **major goal**: prove lower bounds on (non-uniform) circuit size for problems in **NP**
  - believe exponential
  - super-polynomial enough for **P** $\neq$ **NP**
  - best bound known: (5-o(1))·n
  - don't even have super-polynomial bounds for problems in **NEXP**

Lower bounds

- lots of work on lower bounds for restricted classes of circuits
  - we'll see two such lower bounds:
    - formulas
    - monotone circuits

Shannon's counting argument

- frustrating fact: *almost all* functions require huge circuits

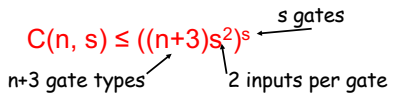
Theorem (Shannon): With probability at least $1 - o(1)$, a random function

$$f:\{0,1\}^n \rightarrow \{0,1\}$$

requires a circuit of size $\Omega(2^n/n)$.

Shannon's counting argument

- Proof (counting):
  - $B(n) = 2^{2^n}$ = # functions $f:\{0,1\}^n \to \{0,1\}$
  - # circuits with n inputs + size s, is at most



Shannon's counting argument

$C(n, s) \le ((n+3)s^2)^s$

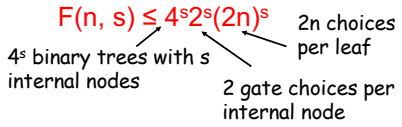
- $C(n, c2^n/n) < ((2n)c^2 2^{2n}/n^2)^{(c2^n/n)}$
  $< o(1)2^{2c2^n}$
  $< o(1)2^{2^n}$ (if $c \le ½$)
- probability a random function has a circuit of size $s = (½)2^n/n$ is at most
  $C(n, s)/B(n) < o(1)$

Shannon's counting argument

Theorem (Shannon): With probability at least $1 - o(1)$, a random function
$f:\{0,1\}^n \to \{0,1\}$
requires a **formula** of size $\Omega(2^n/\log n)$.

Shannon's counting argument

- Proof (counting):
  - $B(n) = 2^{2^n}$ = # functions $f:\{0,1\}^n \to \{0,1\}$
  - # formulas with n inputs + size s, is at most



Shannon's counting argument

$F(n, s) \le 4^s 2^s (2n)^s$

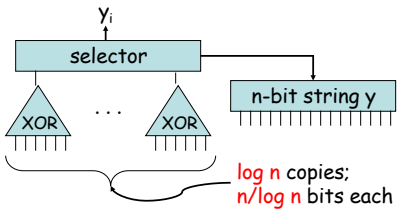
- $F(n, c2^n/\log n) < (16n)^{(c2^n/\log n)}$
  $< 16^{(c2^n/\log n)} 2^{(c2^n)} = (1 + o(1))2^{(c2^n)}$
  $< o(1)2^{2^n}$ (if $c \le ½$)
- probability a random function has a **formula** of size $s = (½)2^n/\log n$ is at most $F(n, s)/B(n) < o(1)$

Andreev function

- best formula lower bound for language in NP:

Theorem (Andreev, Hastad '93): the Andreev function requires $(\wedge,\vee,\neg)$-formulas of size at least
$\Omega(n^{3-o(1)})$.

Andreev function



the Andreev function $A(x,y)$
$A:\{0,1\}^{2n} \to \{0,1\}$

Random restrictions

- key idea: given function
  $f:\{0,1\}^n \to \{0,1\}$
  restrict by ρ to get f_ρ
  - ρ sets some variables to 0/1, others remain free
- $R(n, \epsilon n)$ = set of restrictions that leave ϵn variables free
- Definition: $L(f)$ = smallest $(\wedge,\vee,\neg)$ formula computing f (measured as leaf-size)

Random restrictions

- observation:
  $E_{\rho \leftarrow R(n, \epsilon n)}[L(f_\rho)] \le \epsilon L(f)$
  - each leaf survives with probability ϵ
- may shrink more…
  - propogate constants

Lemma (Hastad 93): for all f
$E_{\rho \leftarrow R(n, \epsilon n)}[L(f_\rho)] \le O(\epsilon^{2-o(1)} L(f))$

Hastad's shrinkage result

- Proof of theorem:
  - Recall: there exists a function
    $h:\{0,1\}^{\log n} \to \{0,1\}$
    for which $L(h) > n/2\log\log n$.
  - hardwire truth table of that function into y to get $A^*(x)$
  - apply random restriction from
    $R(n, m = 2(\log n)(\ln \log n))$
    to $A^*(x)$.

The lower bound

- Proof of theorem (continued):
  - probability given XOR is killed by restriction is probability that we "miss it" m times:
    $(1 - (n/\log n)/n)^m \le (1 - 1/\log n)^m$
    $\le (1/e)^{2\ln \log n} \le 1/\log^2 n$
  - probability even one of XORs is killed by restriction is at most:
    $\log n(1/\log^2 n) = 1/\log n < ½.$

The lower bound

- (1): probability even one of XORs is killed by restriction is at most:
  $\log n(1/\log^2 n) = 1/\log n < ½.$
- (2): by Markov:
  $\Pr[\, L(A^*_\rho) > 2\, E_{\rho \leftarrow R(n, m)}[L(A^*_\rho)] \,] < ½.$
- Conclude: for *some* restriction ρ
  - all XORs survive, and
  - $L(A^*_\rho) \le 2\, E_{\rho \leftarrow R(n, m)}[L(A^*_\rho)]$

The lower bound

- Proof of theorem (continued):
  - if all XORs survive, can restrict formula further to compute hard function h
    - may need to add ¬'s

$$L(h) = \mathbf{n}/2\log\log n \le L(A^*_\rho)$$

$$\le 2E_{\rho \leftarrow R(n,\, m)}[L(A^*_\rho)] \le O((m/n)^{2-o(1)} L(A^*))$$

$$\le O(\, ((\log n)(\ln \log n)/\mathbf{n})^{\mathbf{2-o(1)}}\, L(A^*)\,)$$

  - implies $\Omega(n^{3-o(1)}) \le L(A^*) \le L(A)$.