CS151
Complexity Theory

Lecture 5

April 13, 2021

Remainder of lecture

- nondeterminism applied to space
- reachability
- two surprises:
  - Savitch's Theorem
  - Immerman/Szelepcsényi Theorem

Nondeterministic space

- **NSPACE(f(n))** = languages decidable by a multi-tape NTM that touches at most f(n) squares of its work tapes *along any computation path*, where n is the input length, and f :**N** → **N**

Nondeterministic space

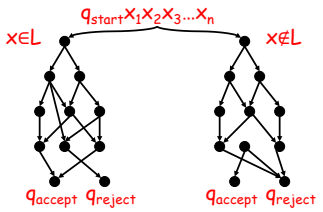
- Robust nondeterministic space classes:

NL = NSPACE(log n)

NPSPACE = $\cup_k$ NSPACE(n^k)

Reachability

- Recall: at most n^k configurations of given NTM M running in **NSPACE**(log n).
  - easy to determine if C yields C' in one step
  - configuration graph for M on input x:



Reachability

- Conclude: **NL ⊆ P**
  - and **NPSPACE ⊆ EXP**
- S-T-Connectivity (STCONN): given directed graph G = (V, E) and nodes s, t, is there a path from s to t ?

Theorem: STCONN is **NL**-complete under logspace reductions.

Reachability

- Proof:
  - in **NL**: guess path from s to t one node at a time
  - given L ∈ **NL** decided by NTM M construct configuration graph for M on input x (can be done in logspace)
  - s = starting configuration; t = q_{accept}

Two startling theorems

- Strongly believe **P ≠ NP**
- nondeterminism seems to add enormous power
- for space: Savitch '70:

NPSPACE = PSPACE

and

NL ⊆ SPACE($\log^2 n$)

Two startling theorems

- Strongly believe **NP ≠ coNP**
- seems impossible to convert existential into universal
- for space: Immerman/Szelepscényi '87/'88:

NL = coNL

Savitch's Theorem

Theorem: STCONN ⊆ **SPACE($\log^2 n$)**

- Corollary: **NL ⊆ SPACE($\log^2 n$)**
- Corollary: **NPSPACE = PSPACE**

Proof of Theorem

- input: G = (V, E), two nodes s and t
- recursive algorithm:

```
/* return true iff path from x to y of length at most 2^i */
PATH(x, y, i)
 if i = 0 return ( x = y or (x, y) ∈ E )        /* base case */
 for z in V
    if PATH(x, z, i-1) and PATH(z, y, i-1) return(true);
 return(false);
end
```


Proof of Theorem

- answer to STCONN: PATH(s, t, log n)
- space used:
  - (depth of recursion) x (size of "stack record")
- depth = log n
- claim stack record: "(x, y, i)" sufficient
  - size O(log n)
- when return from PATH(a, b, i) can figure out what to do next from record (a, b, i) and previous record

Nondeterministic space

- Robust nondeterministic space classes:

NL = NSPACE(log n)

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Second startling theorem

- Strongly believe **NP ≠ coNP**
- seems impossible to convert existential into universal
- for space: Immerman/Szelepscényi '87/'88:

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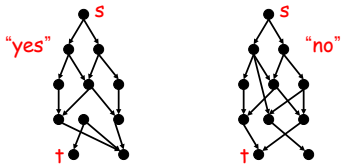
I-S Theorem

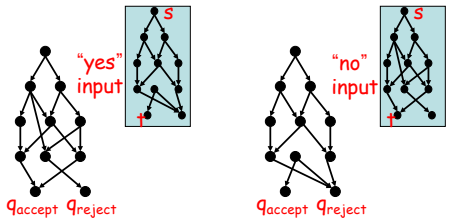
Theorem: ST-NON-CONN ∈ **NL**

- Proof: slightly tricky setup:
  - input: G = (V, E), two nodes s, t



I-S Theorem

- want nondeterministic procedure using only O(log n) space with behavior:



I-S Theorem

- observation: given **count** of # nodes reachable from s, can solve problem
  - for each v ∈ V, *guess* if it is reachable
  - if yes, *guess* path from s to v
    - if guess doesn't lead to v, reject.
    - if v = t, reject.
    - else counter++
  - if counter = **count** accept

I-S Theorem

- every computation path has sequence of guesses…
- only way computation path can lead to accept:
  - correctly guessed reachable/unreachable for each node v
  - correctly guessed path from s to v for each reachable node v
  - saw *all* reachable nodes
  - t not among reachable nodes

I-S Theorem

– R(i) = # nodes reachable from s in at most i steps

– R(0) = 1: node s

– we will compute R(i+1) from R(i) using O(log n) space and nondeterminism

– computation paths with "bad guesses" all lead to reject

I-S Theorem

– Outline: in n phases, compute

R(1), R(2), R(3), … R(n)

– only O(log n) bits of storage between phases

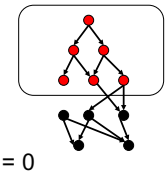
– in end, lots of computation paths that lead to reject

– only computation paths that survive have computed correct value of R(n)

– apply observation.

I-S Theorem

– computing R(i+1) from R(i):



R(i) = R(2) = 6

– Initialize R(i+1) = 0

– For each v ∈ V, *guess* if v reachable from s in at most i+1 steps

I-S Theorem

– if "yes", *guess* path from s to v of at most i+1 steps. Increment R(i+1)

– if "no", visit R(i) nodes reachable in at most i steps, check that none is v or adjacent to v

- for u ∈ V *guess* if reachable in ≤ i steps; *guess* path to u; counter++
- KEY: if counter ≠ R(i), reject
- at this point: **can be sure v not reachable**

I-S Theorem

- correctness of procedure:
- two types of errors we can make
- (1) might guess v is reachable in at most i+1 steps when it is not
  – won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error

I-S Theorem

- (2) might guess v is **not** reachable in at most i+1 steps when it is
  – then must **not** see v or neighbor of v while visiting nodes reachable in i steps.
  – but forced to visit R(i) distinct nodes
  – therefore must try to visit node v that is **not** reachable in ≤ i steps
  – won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error

Summary

- nondeterministic space classes
  NL and **NPSPACE**
- ST-CONN **NL**-complete

Summary

- Savitch: **NPSPACE = PSPACE**
  – Proof: ST-CONN ∈ **SPACE($\log^2 n$)**
  – open question:
  NL = L?
- Immerman/Szelepcsényi : **NL = coNL**
  – Proof: ST-NON-CONN ∈ **NL**

Introduction

Power from an unexpected source?

- we know **P ≠ EXP**, which implies no poly-time *algorithm* for Succinct CVAL
- poly-size Boolean *circuits* for Succinct CVAL ??

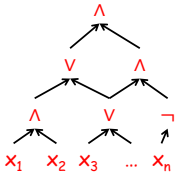
Does **NP** have linear-size, log-depth Boolean circuits ??

Outline

- Boolean circuits and formulas
- uniformity and advice
- the **NC** hierarchy and parallel computation
- the quest for circuit lower bounds
- a lower bound for formulas

Boolean circuits

- **circuit** C
  – directed acyclic graph
  – nodes: AND (∧); OR (∨); NOT (¬); variables x_i



- C computes function $f:\{0,1\}^n \to \{0,1\}$ in natural way
  – identify C with function f it computes

Boolean circuits

- **size** = # gates
- **depth** = longest path from input to output
- **formula (or expression)**: graph is a tree

- every function $f:\{0,1\}^n \to \{0,1\}$ computable by a circuit of size at most $O(n2^n)$
  – AND of n literals for each x such that f(x) = 1
  – OR of up to 2^n such terms

Circuit families

- circuit works for specific input length
- we're used to $f:\sum^* \rightarrow \{0,1\}$
- circuit **family** : a circuit for each input length $C_1, C_2, C_3, \ldots$ = "$\{C_n\}$"
- "$\{C_n\}$ computes f" iff for all x

$$C_{|x|}(x) = f(x)$$

- "$\{C_n\}$ decides L", where L is the language associated with f

Connection to TMs

- given TM M running in time t(n) decides language L
- can build circuit family $\{C_n\}$ that decides L
  - size of $C_n = O(t(n)^2)$
  - Proof: CVAL construction
- Conclude: L ∈ **P** implies family of polynomial-size circuits that decides L

Connection to TMs

- other direction?
- A poly-size circuit family:
  - $C_n = (x_1 \vee \neg x_1)$ if M_n halts
  - $C_n = (x_1 \wedge \neg x_1)$ if M_n loops
- decides (unary version of) HALT!
- oops…

Uniformity

- Strange aspect of circuit family:
  - can "encode" (potentially uncomputable) information in family specification
- solution: uniformity – require specification is simple to compute

Definition: circuit family $\{C_n\}$ is **logspace uniform** iff TM M outputs C_n on input 1^n and runs in O(log n) space

Uniformity

Theorem: **P** = languages decidable by logspace uniform, polynomial-size circuit families $\{C_n\}$.

- Proof:
  - already saw (⇒)
  - (⇐) on input x, generate $C_{|x|}$, evaluate it and accept iff output = 1

TMs that take advice

- family $\{C_n\}$ without uniformity constraint is called "non-uniform"
- regard "non-uniformity" as a limited resource just like time, space, as follows:
  - add read-only "advice" tape to TM M
  - M "decides L with advice A(n)" iff

$$M(x, A(|x|)) \text{ accepts} \Leftrightarrow x \in L$$

  - note: A(n) depends only on |x|

TMs that take advice

- Definition: **TIME(t(n))/f(n)** = the set of those languages L for which:
  - there exists A(n) s.t. $|A(n)| \leq f(n)$
  - TM M decides L with advice A(n) in time t(n)
- most important such class:

$$\mathbf{P/poly} = \cup_k \mathbf{TIME}(n^k)/n^k$$

TMs that take advice

Theorem: L ∈ **P/poly** iff L decided by family of (non-uniform) polynomial size circuits.

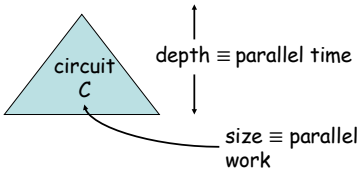
- Proof:
  - (⇒) C_n from CVAL construction; hardwire advice A(n)
  - (⇐) define A(n) = description of C_n; on input x, TM simulates $C_{|x|}(x)$

Approach to P/NP

- Believe **NP** ≠ **P**
  - equivalent: "**NP** does not have uniform, polynomial-size circuits"
- *Even believe* **NP** ⊄ **P/poly**
  - equivalent: "**NP** (or, e.g. SAT) does not have polynomial-size circuits"
  - implies **P** ≠ **NP**
  - many believe: best hope for **P** ≠ **NP**

Parallelism

- uniform circuits allow refinement of polynomial time:



Parallelism

- the **NC** ("**N**ick's **C**lass") Hierarchy (of logspace uniform circuits):

$$\mathbf{NC}_k = O(\log^k n) \text{ depth, poly}(n) \text{ size}$$

$$\mathbf{NC} = \cup_k \mathbf{NC}_k$$

- captures "efficiently parallelizable problems"
- not realistic? overly generous
- OK for proving non-parallelizable