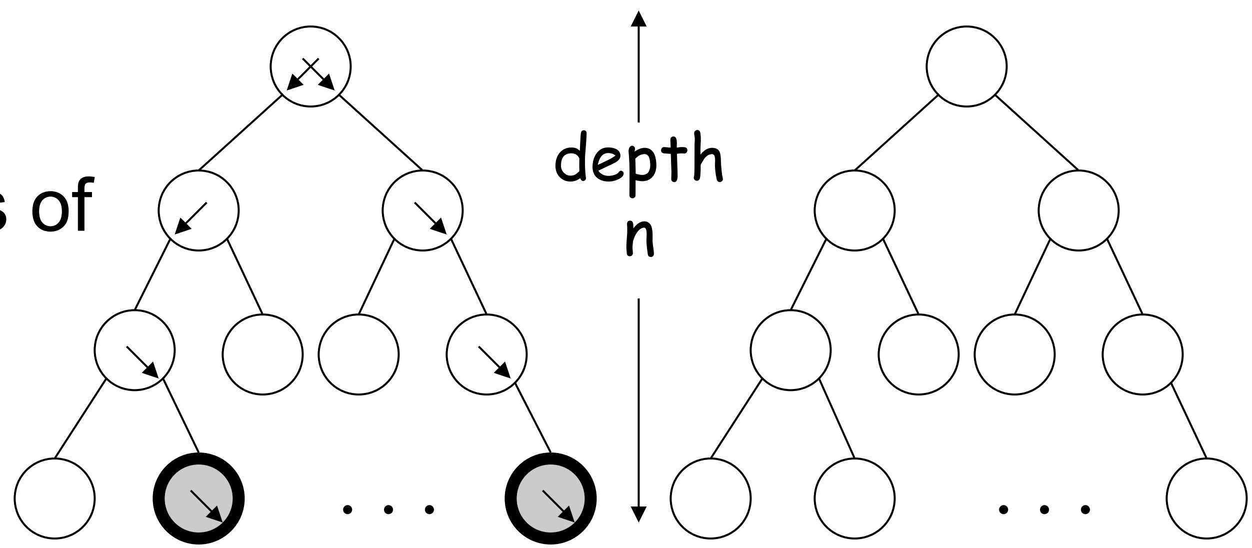
CS151
Complexity Theory

Lecture 4

April 12, 2017

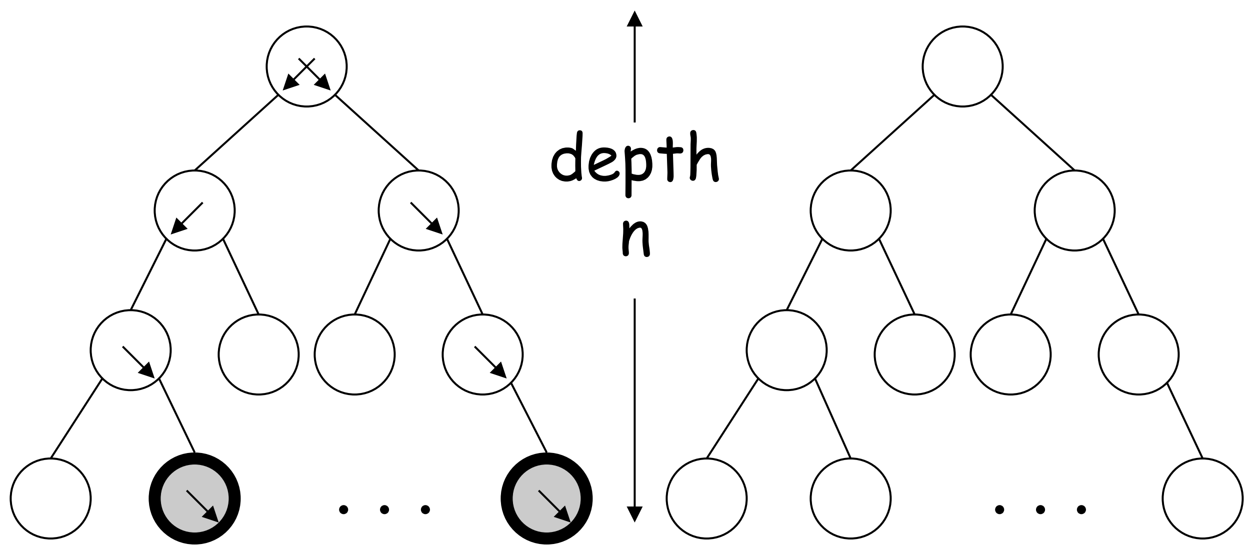
A puzzle

A puzzle:
two kinds of
trees

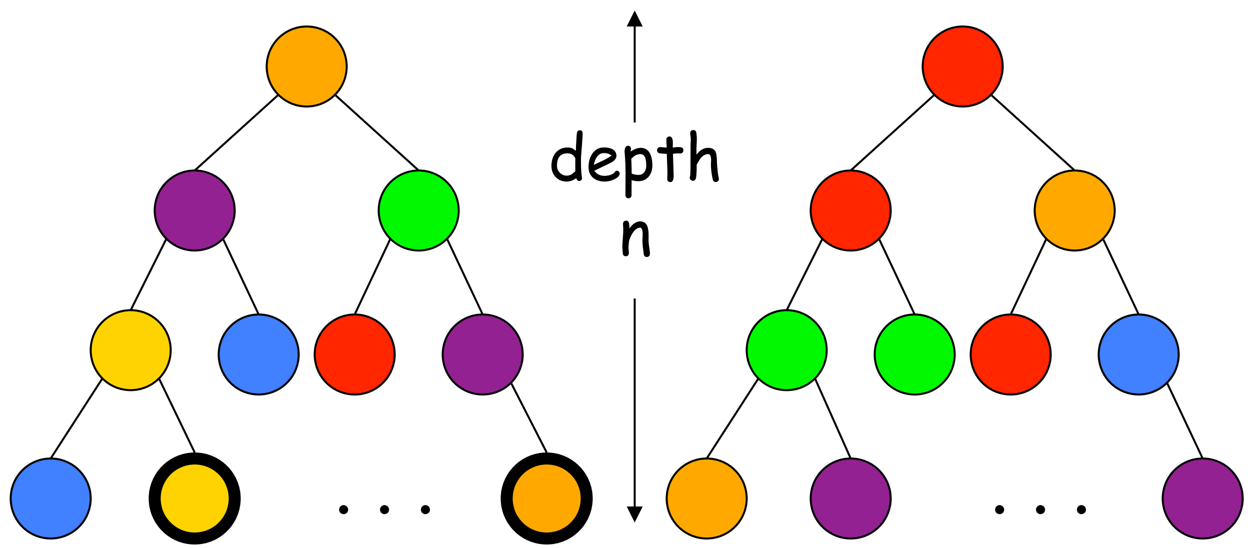


- cover up nodes with c colors
- promise: never color "arrow" same as "blank"
- determine which kind of tree in poly(n, c) steps?

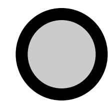
A puzzle



A puzzle



Introduction

- Ideas
  - depth-first-search; stop if see 
  - how many times may we see a given “arrow color”?
    - at most n+1
  - pruning rule?
    - if see a color > n+1 times, it can’t be an arrow node; prune
  - # nodes visited before know answer?
    - at most c(n+2)

Sparse languages and **NP**

- We often say **NP**-compete languages are "hard"

- More accurate: **NP**-complete languages are "expressive"
  - lots of languages reduce to them

Sparse languages and **NP**

- Sparse language: one that contains at most poly(n) strings of length ≤ n
- not very expressive – can we show this cannot be **NP**-complete (assuming **P** ≠ **NP**) ?
  - yes: Mahaney ’82 (homework problem)
- Unary language: subset of 1* (at most n strings of length ≤ n)

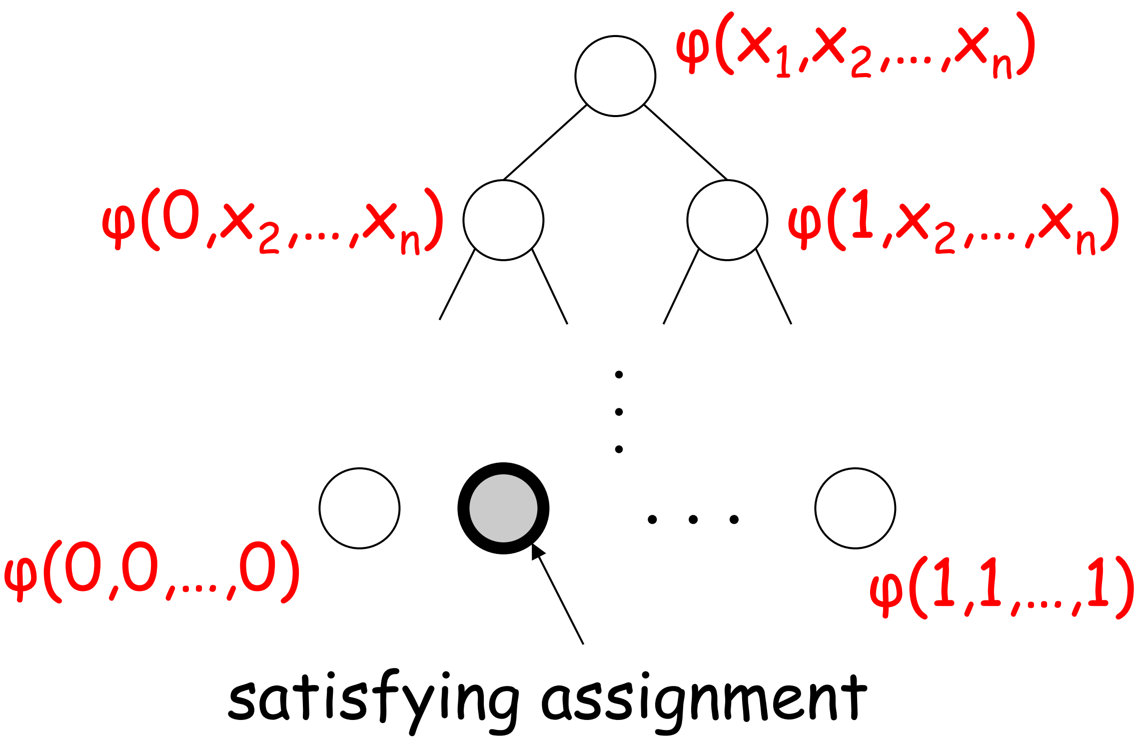
Sparse languages and **NP**

<u>**Theorem**</u> (Berman '78): if a unary language is **NP**-complete then **P** = **NP**.

- Proof:
  - let $U \subset 1^*$ be a unary language and assume SAT $\leq$ U via reduction R
  - $\varphi(x_1,x_2,\ldots,x_n)$ instance of SAT

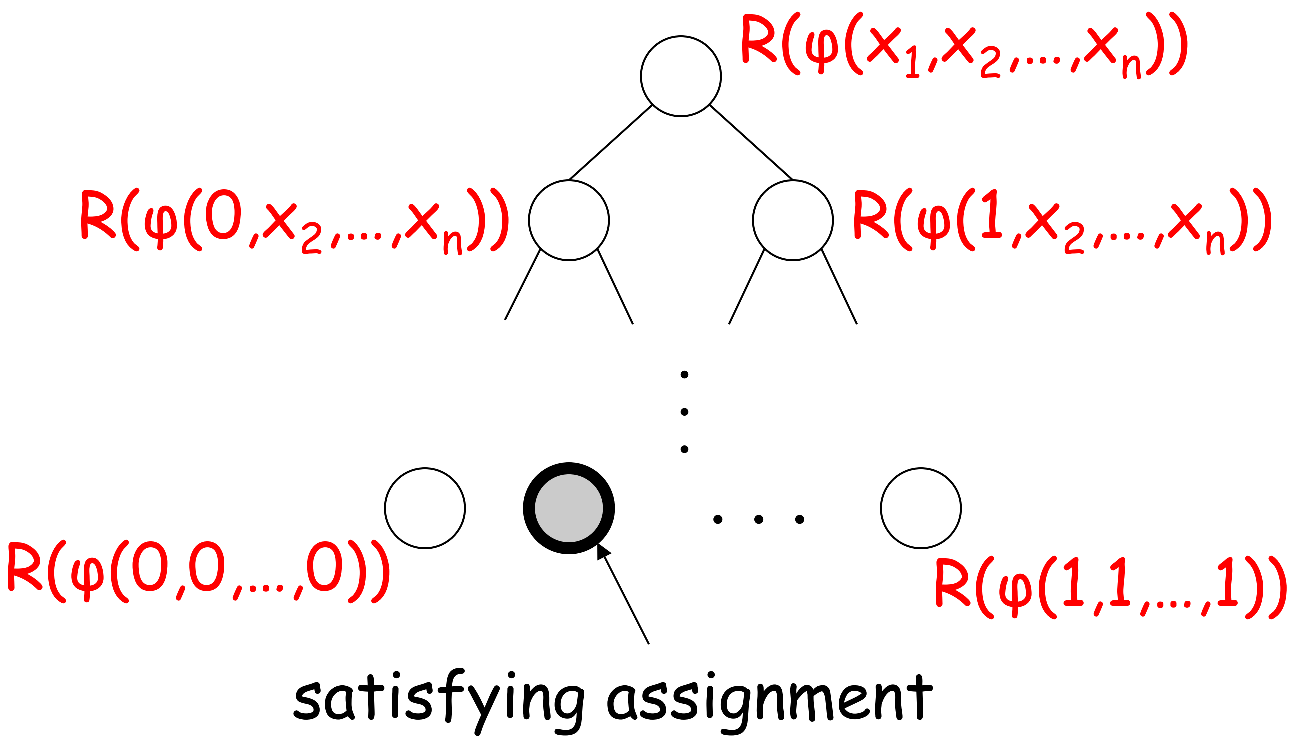
Sparse languages and **NP**

- self-reduction tree for φ:



Sparse languages and **NP**

– applying reduction R:



Sparse languages and **NP**

- on input of length $m = |\varphi(x_1,x_2,\ldots,x_n)|$, R produces string of length $\leq p(m)$
- R's different outputs are "colors"
  - 1 color for strings not in 1^*
  - at most p(m) other colors

- puzzle solution $\Rightarrow$ can solve SAT in poly(p(m)+1, n+1) = poly(m) time!

Summary

- nondeterministic time classes:

  NP, coNP, NEXP

- NTIME Hierarchy Theorem:

  NP $\neq$ NEXP

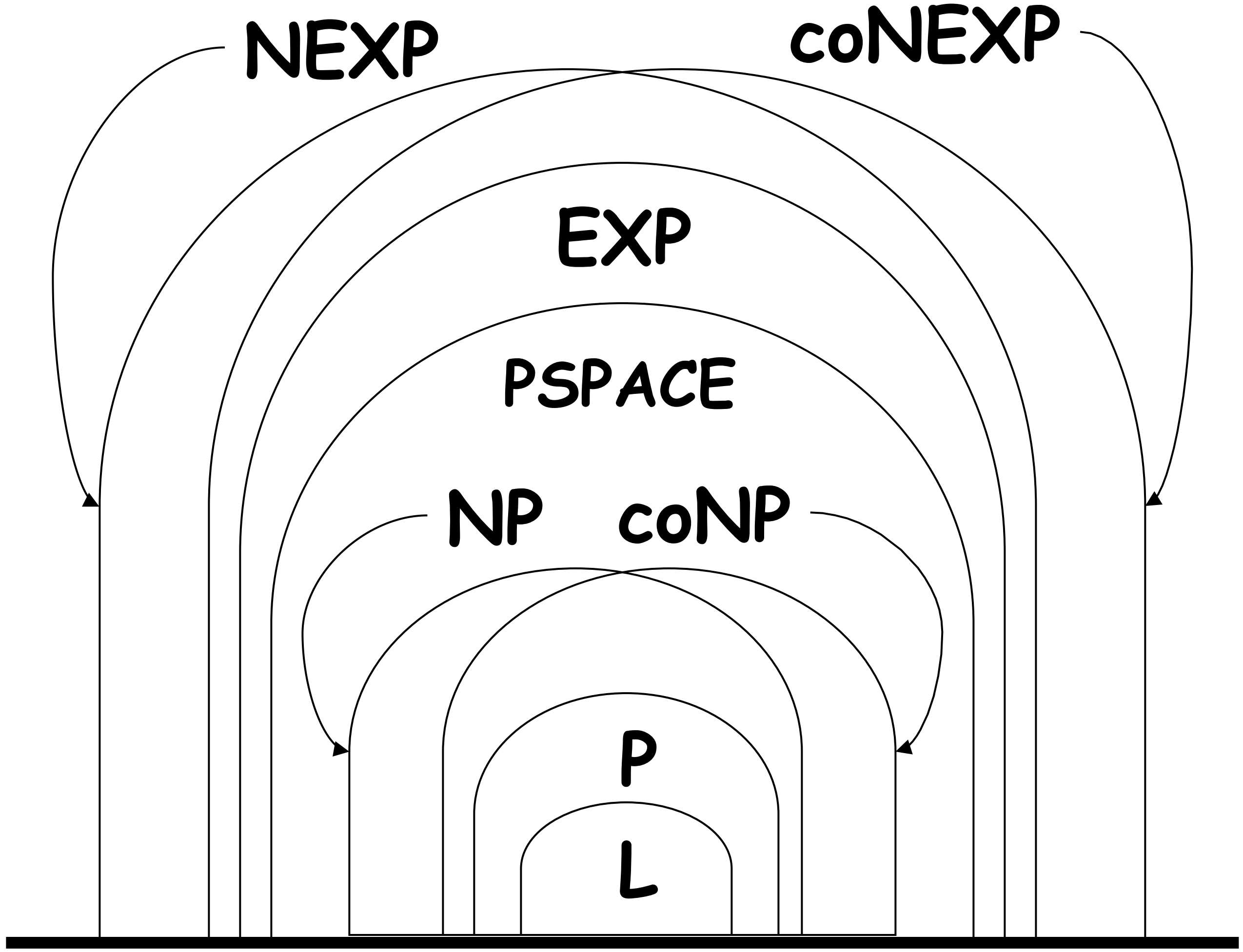
- major open questions:

  $$\mathbf{P} \stackrel{?}{=} \mathbf{NP} \qquad \mathbf{NP} \stackrel{?}{=} \mathbf{coNP}$$

Summary

- **NP**-"intermediate" problems (unless **P** = **NP**)
  - technique: delayed diagonalization
- unary languages not **NP**-complete (unless **P** = **NP**)
  - true for sparse languages as well (homework)
- complete problems:
  - circuit SAT is **NP**-complete
  - UNSAT is **coNP**-complete
  - succinct circuit SAT is **NEXP**-complete

Summary



Remainder of lecture

- nondeterminism applied to space
- reachability
- two surprises:
  - Savitch's Theorem
  - Immerman/Szelepcsényi Theorem

Nondeterministic space

- **NSPACE(f(n))** = languages decidable by a multi-tape NTM that touches at most f(n) squares of its work tapes *along any computation path*, where n is the input length, and f :$\mathbf{N} \rightarrow \mathbf{N}$

Nondeterministic space

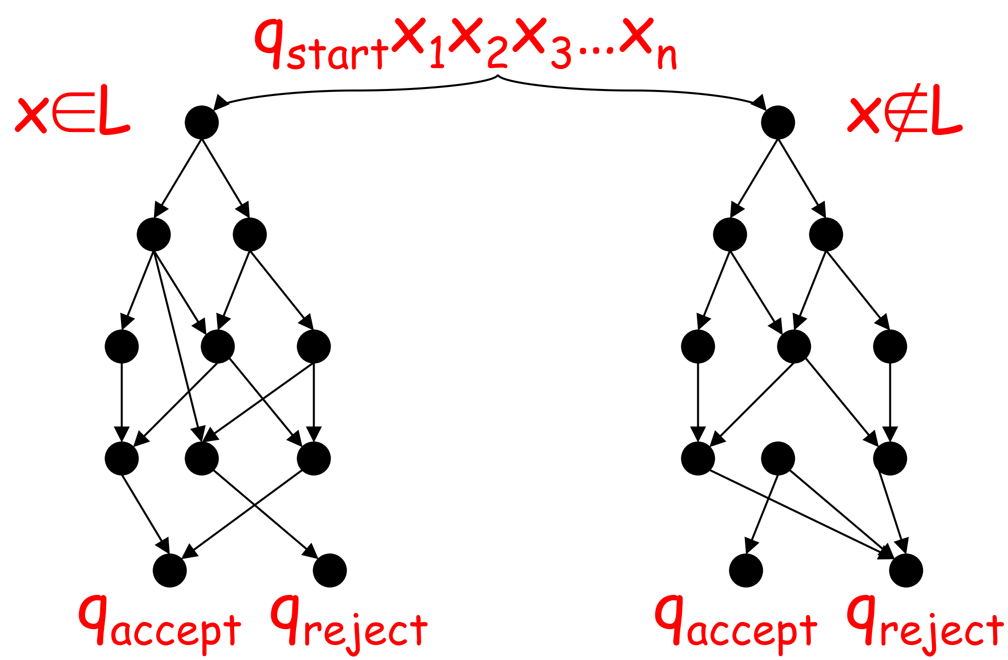
- Robust nondeterministic space classes:

$$\mathbf{NL} = \mathbf{NSPACE}(\log n)$$

$$\mathbf{NPSPACE} = \cup_k \mathbf{NSPACE}(n^k)$$

Reachability

- Recall: at most n^k configurations of given NTM M running in **NSPACE**(log n).
  - easy to determine if C yields C' in one step
  - configuration graph for M on input x:



Reachability

- Conclude: **NL $\subset$ P**
  - and **NPSPACE $\subset$ EXP**

- S-T-Connectivity (STCONN): given directed graph G = (V, E) and nodes s, t, is there a path from s to t ?

Theorem: STCONN is **NL**-complete under logspace reductions.

Reachability

- Proof:
  - in **NL**: guess path from s to t one node at a time
  - given L $\in$ **NL** decided by NTM M construct configuration graph for M on input x (can be done in logspace)
  - s = starting configuration; t = q_{accept}

Two startling theorems

- Strongly believe **P ≠ NP**
- nondeterminism seems to add enormous power
- for space: Savitch '70:

NPSPACE = PSPACE

and

NL ⊂ SPACE($\log^2 n$)

Two startling theorems

- Strongly believe **NP ≠ coNP**
- seems impossible to convert existential into universal

- for space: Immerman/Szelepscényi '87/'88:

NL = coNL

Savitch's Theorem

<u>Theorem</u>: STCONN $\in$ **SPACE($\log^2$ n)**

- Corollary: **NL $\subset$ SPACE($\log^2$n)**
- Corollary: **NPSPACE = PSPACE**

Proof of Theorem

– input: G = (V, E), two nodes s and t
– recursive algorithm:

```
/* return true iff path from x to y of length at most 2^i */
PATH(x, y, i)
 if i = 0 return ( x = y or (x, y) ∈ E )          /* base case */
 for z in V
     if PATH(x, z, i-1) and PATH(z, y, i-1) return(true);
 return(false);
end
```

Proof of Theorem

– answer to STCONN: PATH(s, t, log n)
– space used:
  - (depth of recursion) x (size of "stack record")
– depth = log n
– claim stack record: "(x, y, i)" sufficient
  - size O(log n)
– when return from PATH(a, b, i) can figure out what to do next from record (a, b, i) and previous record

Nondeterministic space

- Robust nondeterministic space classes:

$$\mathbf{NL} = \mathbf{NSPACE}(\log n)$$

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Second startling theorem

- Strongly believe **NP ≠ coNP**
- seems impossible to convert existential into universal

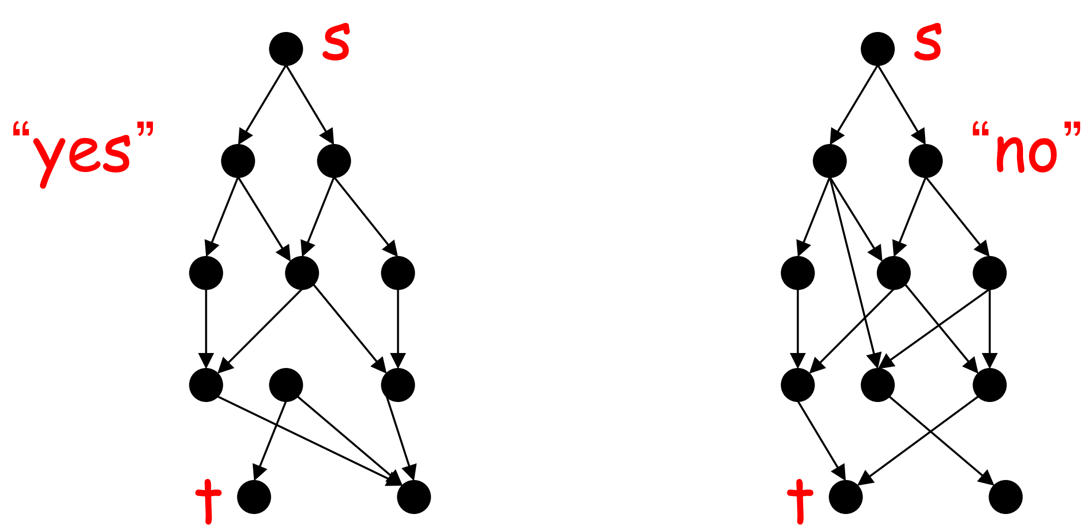
- for space: Immerman/Szelepscényi ’87/’88:

NL = coNL

I-S Theorem

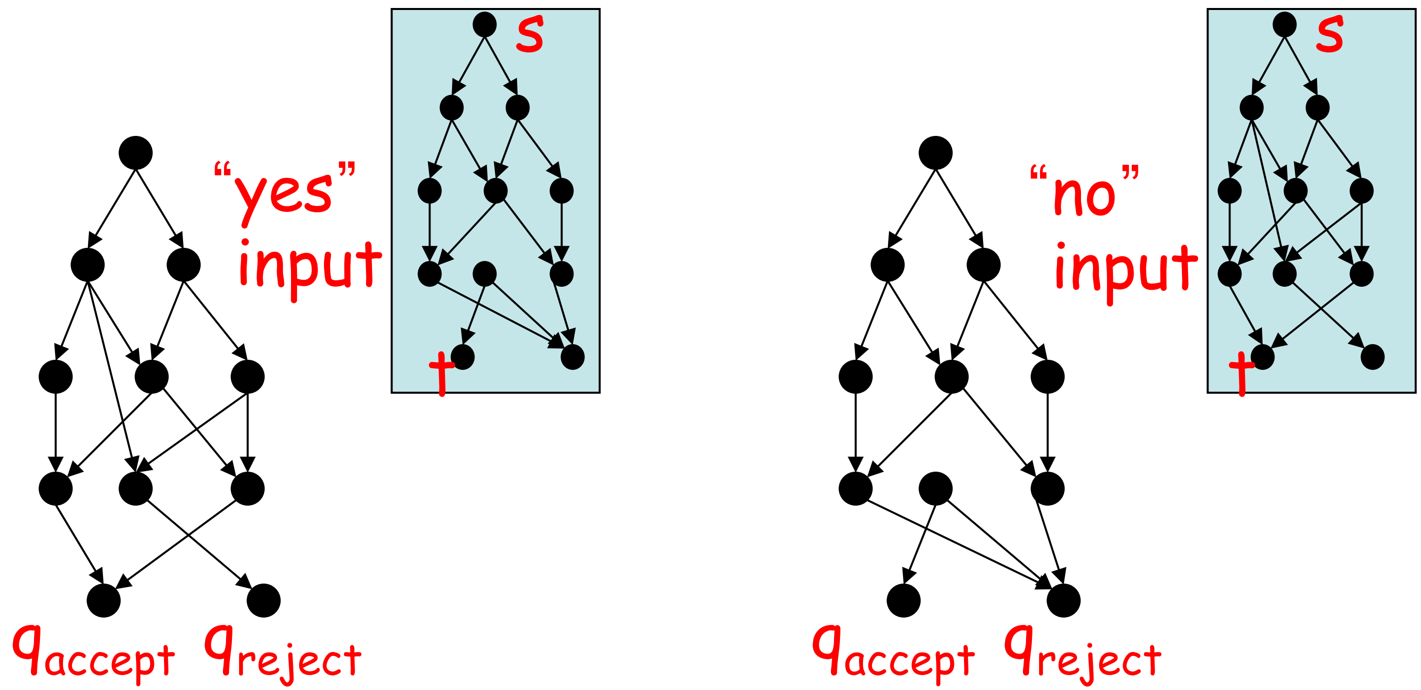
<u>Theorem</u>: ST-NON-CONN $\in$ **NL**

- Proof: slightly tricky setup:
  - input: G = (V, E), two nodes s, t



I-S Theorem

– want nondeterministic procedure using only O(log n) space with behavior:



I-S Theorem

- observation: given **count** of # nodes reachable from s, can solve problem
  - for each $v \in V$, *guess* if it is reachable
  - if yes, *guess* path from s to v
    - if guess doesn't lead to v, reject.
    - if v = t, reject.
    - else counter++
  - if counter = **count** accept

I-S Theorem

- every computation path has sequence of guesses…
- only way computation path can lead to accept:
  - correctly guessed reachable/unreachable for each node v
  - correctly guessed path from s to v for each reachable node v
  - saw *all* reachable nodes
  - t not among reachable nodes

I-S Theorem

- R(i) = # nodes reachable from s in at most i steps
- R(0) = 1: node s
- we will compute R(i+1) from R(i) using O(log n) space and nondeterminism
- computation paths with "bad guesses" all lead to reject

I-S Theorem

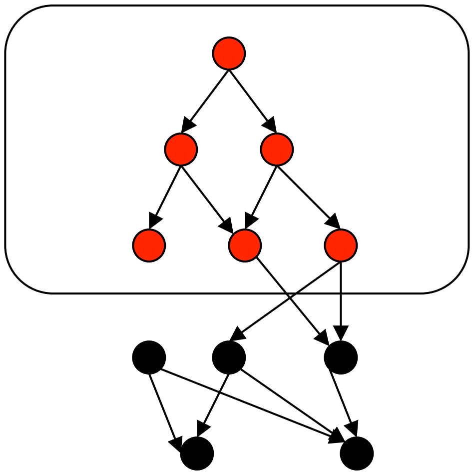
- Outline: in n phases, compute

  R(1), R(2), R(3), … R(n)

- only O(log n) bits of storage between phases
- in end, lots of computation paths that lead to reject
- only computation paths that survive have computed correct value of R(n)
- apply observation.

I-S Theorem

– computing R(i+1) from R(i):



R(i) = R(2) = 6

– Initialize R(i+1) = 0

– For each v ∈ V, *guess* if v reachable from s in at most i+1 steps

I-S Theorem

- if "yes", *guess* path from s to v of at most i+1 steps. Increment R(i+1)
- if "no", visit R(i) nodes reachable in at most i steps, check that none is v or adjacent to v
  - for u $\in$ V *guess* if reachable in $\leq$ i steps; *guess* path to u; counter++
  - KEY: if counter $\neq$ R(i), reject
  - at this point: **can be sure v not reachable**

I-S Theorem

- correctness of procedure:
- two types of errors we can make
- (1) might guess v is reachable in at most i +1 steps when it is not
  - won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error

I-S Theorem

- (2) might guess v is **not** reachable in at most i+1 steps when it is
  - then must **not** see v or neighbor of v while visiting nodes reachable in i steps.
  - but forced to visit R(i) distinct nodes
  - therefore must try to visit node v that is **not** reachable in ≤ i steps
  - won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error

Summary

- nondeterministic space classes

  NL and **NPSPACE**

- ST-CONN **NL**-complete

Summary

- Savitch: **NPSPACE = PSPACE**
  - Proof: ST-CONN $\in$ **SPACE($\log^2 n$)**
  - open question:

    NL = L?

- Immerman/Szelepcsényi : **NL = coNL**
  - Proof: ST-NON-CONN $\in$ **NL**