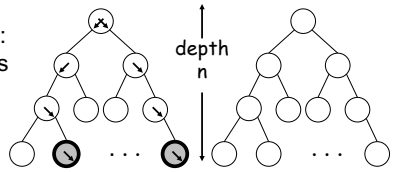
CS151
Complexity Theory

Lecture 4

April 12, 2019

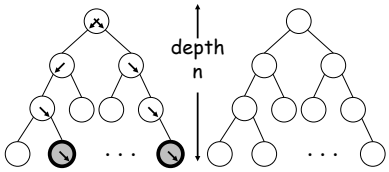
A puzzle

A puzzle: two kinds of trees

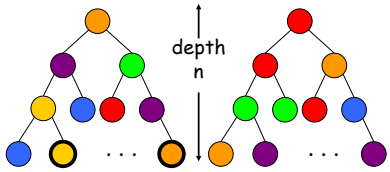


- cover up nodes with c colors
- promise: never color "arrow" same as "blank"
- determine which kind of tree in poly(n, c) steps?

A puzzle



A puzzle



Introduction

- Ideas
  - depth-first-search; stop if see
  - how many times may we see a given "arrow color"?
    - at most n+1
  - pruning rule?
    - if see a color > n+1 times, it can't be an arrow node; prune
  - # nodes visited before know answer?
    - at most c(n+2)

Sparse languages and **NP**

- We often say **NP**-compete languages are "hard"
- More accurate: **NP**-complete languages are "expressive"
  - lots of languages reduce to them

Sparse languages and **NP**

- Sparse language: one that contains at most poly(n) strings of length ≤ n
- not very expressive – can we show this cannot be **NP**-complete (assuming **P** ≠ **NP**) ?
  - yes: Mahaney '82 (homework problem)
- Unary language: subset of 1* (at most n strings of length ≤ n)

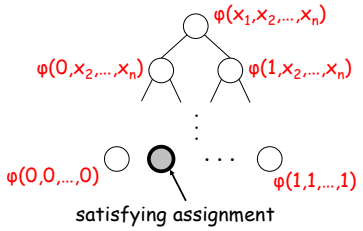
Sparse languages and **NP**

Theorem (Berman '78): if a unary language is **NP**-complete then **P** = **NP**.

- Proof:
  - let U ⊆ 1* be a unary language and assume SAT ≤ U via reduction R
  - $\varphi(x_1,x_2,\ldots,x_n)$ instance of SAT

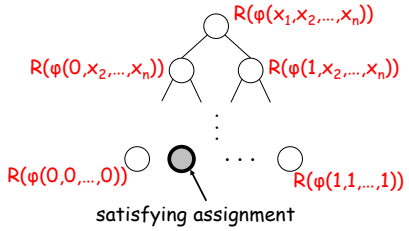
Sparse languages and **NP**

– self-reduction tree for φ:



Sparse languages and **NP**

– applying reduction R:



Sparse languages and **NP**

- on input of length $m = |\varphi(x_1,x_2,\ldots,x_n)|$, R produces string of length ≤ p(m)
- R's different outputs are "colors"
  - 1 color for strings not in 1*
  - at most p(m) other colors
- puzzle solution ⇒ can solve SAT in poly(p(m)+1, n+1) = poly(m) time!

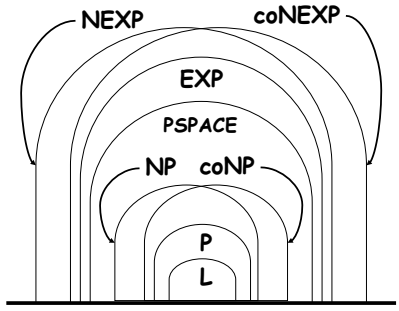
Summary

- nondeterministic time classes:
  NP, coNP, NEXP
- NTIME Hierarchy Theorem:
  NP ≠ NEXP
- major open questions:
  P $\stackrel{?}{=}$ NP **NP $\stackrel{?}{=}$ coNP**

Summary

- **NP**-"intermediate" problems (unless **P** = **NP**)
  - technique: delayed diagonalization
- unary languages not **NP**-complete (unless **P** = **NP**)
  - true for sparse languages as well (homework)
- complete problems:
  - circuit SAT is **NP**-complete
  - UNSAT is **coNP**-complete
  - succinct circuit SAT is **NEXP**-complete

Summary



Remainder of lecture

- nondeterminism applied to space
- reachability
- two surprises:
  - Savitch's Theorem
  - Immerman/Szelepcsényi Theorem

Nondeterministic space

- **NSPACE(f(n))** = languages decidable by a multi-tape NTM that touches at most f(n) squares of its work tapes *along any computation path*, where n is the input length, and f :**N** → **N**

Nondeterministic space

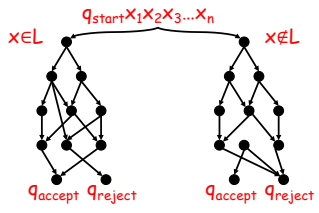
- Robust nondeterministic space classes:

NL = NSPACE(log n)

NPSPACE = $\cup_k$ NSPACE(n^k)

Reachability

- Recall: at most n^k configurations of given NTM M running in **NSPACE**(log n).
  - easy to determine if C yields C' in one step
  - configuration graph for M on input x:



Reachability

- Conclude: **NL ⊆ P**
  - and **NPSPACE ⊆ EXP**
- S-T-Connectivity (STCONN): given directed graph G = (V, E) and nodes s, t, is there a path from s to t ?

Theorem: STCONN is **NL**-complete under logspace reductions.

Reachability

- Proof:
  - in **NL**: guess path from s to t one node at a time
  - given L ∈ **NL** decided by NTM M construct configuration graph for M on input x (can be done in logspace)
  - s = starting configuration; t = q_{accept}

Two startling theorems

- Strongly believe **P ≠ NP**
- nondeterminism seems to add enormous power
- for space: Savitch '70:

NPSPACE = PSPACE

and

NL ⊆ SPACE($\log^2 n$)

Two startling theorems

- Strongly believe **NP ≠ coNP**
- seems impossible to convert existential into universal
- for space: Immerman/Szelepscényi '87/'88:

NL = coNL

Savitch's Theorem

Theorem: STCONN ⊆ **SPACE($\log^2 n$)**

- Corollary: **NL ⊆ SPACE($\log^2 n$)**
- Corollary: **NPSPACE = PSPACE**

Proof of Theorem

- input: G = (V, E), two nodes s and t
- recursive algorithm:

```
/* return true iff path from x to y of length at most 2^i */
PATH(x, y, i)
 if i = 0 return ( x = y or (x, y) ∈ E )        /* base case */
 for z in V
    if PATH(x, z, i-1) and PATH(z, y, i-1) return(true);
 return(false);
end
```

Proof of Theorem

– answer to STCONN: PATH(s, t, log n)
– space used:
  • (depth of recursion) x (size of "stack record")
– depth = log n
– claim stack record: "(x, y, i)" sufficient
  • size O(log n)
– when return from PATH(a, b, i) can figure out what to do next from record (a, b, i) and previous record

Nondeterministic space

• Robust nondeterministic space classes:

NL = NSPACE(log n)

NPSPACE = $\cup_k$ NSPACE(n^k)

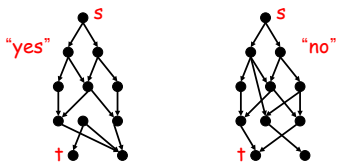
Second startling theorem

• Strongly believe **NP ≠ coNP**
• seems impossible to convert existential into universal
• for space: Immerman/Szelepscényi '87/'88:

NL = coNL

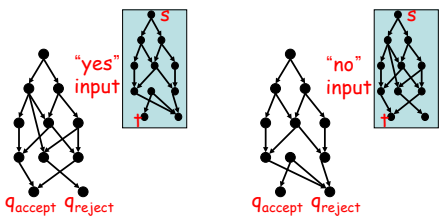
I-S Theorem

Theorem: ST-NON-CONN ∈ **NL**
• Proof: slightly tricky setup:
  – input: G = (V, E), two nodes s, t



I-S Theorem

– want nondeterministic procedure using only O(log n) space with behavior:



I-S Theorem

– observation: given **count** of # nodes reachable from s, can solve problem
  • for each v ∈ V, *guess* if it is reachable
  • if yes, *guess* path from s to v
    – if guess doesn't lead to v, reject.
    – if v = t, reject.
    – else counter++
  • if counter = **count** accept

I-S Theorem

- every computation path has sequence of guesses…
- only way computation path can lead to accept:
  - correctly guessed reachable/unreachable for each node v
  - correctly guessed path from s to v for each reachable node v
  - saw *all* reachable nodes
  - t not among reachable nodes

I-S Theorem

- R(i) = # nodes reachable from s in at most i steps
- R(0) = 1: node s
- we will compute R(i+1) from R(i) using O(log n) space and nondeterminism
- computation paths with "bad guesses" all lead to reject

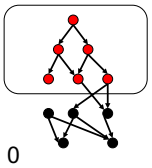
I-S Theorem

- Outline: in n phases, compute

  R(1), R(2), R(3), … R(n)
- only O(log n) bits of storage between phases
- in end, lots of computation paths that lead to reject
- only computation paths that survive have computed correct value of R(n)
- apply observation.

I-S Theorem

- computing R(i+1) from R(i):



- Initialize R(i+1) = 0
- For each v ∈ V, *guess* if v reachable from s in at most i+1 steps

I-S Theorem

- if "yes", *guess* path from s to v of at most i+1 steps. Increment R(i+1)
- if "no", visit R(i) nodes reachable in at most i steps, check that none is v or adjacent to v
  - for u ∈ V *guess* if reachable in ≤ i steps; *guess* path to u; counter++
  - KEY: if counter ≠ R(i), reject
  - at this point: **can be sure v not reachable**

I-S Theorem

- correctness of procedure:
- two types of errors we can make
- (1) might guess v is reachable in at most i+1 steps when it is not
  - won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error

I-S Theorem

- (2) might guess v is **not** reachable in at most i+1 steps when it is
  - then must **not** see v or neighbor of v while visiting nodes reachable in i steps.
  - but forced to visit R(i) distinct nodes
  - therefore must try to visit node v that is **not** reachable in ≤ i steps
  - won't be able to guess path from s to v of correct length, so we will reject.

"easy" type of error

Summary

- nondeterministic space classes
  NL and **NPSPACE**
- ST-CONN **NL**-complete

Summary

- Savitch: **NPSPACE = PSPACE**
  - Proof: ST-CONN ∈ **SPACE($\log^2 n$)**
  - open question:
    NL = L?
- Immerman/Szelepcsényi : **NL = coNL**
  - Proof: ST-NON-CONN ∈ **NL**

Introduction

Power from an unexpected source?

- we know **P ≠ EXP**, which implies no poly-time *algorithm* for Succinct CVAL
- poly-size Boolean *circuits* for Succinct CVAL ??

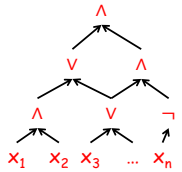
Does **NP** have linear-size, log-depth Boolean circuits ??

Outline

- Boolean circuits and formulas
- uniformity and advice
- the **NC** hierarchy and parallel computation
- the quest for circuit lower bounds
- a lower bound for formulas

Boolean circuits

- **circuit** C
  - directed acyclic graph
  - nodes: AND (∧); OR (∨); NOT (¬); variables x_i



- C computes function $f:\{0,1\}^n \rightarrow \{0,1\}$ in natural way
  - identify C with function f it computes

Boolean circuits

- **size** = # gates
- **depth** = longest path from input to output
- **formula (or expression)**: graph is a tree

- every function $f:\{0,1\}^n \rightarrow \{0,1\}$ computable by a circuit of size at most $O(n2^n)$
  - AND of n literals for each x such that f(x) = 1
  - OR of up to 2^n such terms

Circuit families

- circuit works for specific input length
- we're used to $f:\sum^* \rightarrow \{0,1\}$
- circuit **family** : a circuit for each input length $C_1, C_2, C_3, \ldots$ = "$\{C_n\}$"
- "$\{C_n\}$ computes f" iff for all x

$$C_{|x|}(x) = f(x)$$

- "$\{C_n\}$ decides L", where L is the language associated with f

Connection to TMs

- given TM M running in time t(n) decides language L
- can build circuit family $\{C_n\}$ that decides L
  - size of $C_n = O(t(n)^2)$
  - Proof: CVAL construction

- Conclude: L ∈ **P** implies family of polynomial-size circuits that decides L

Connection to TMs

- other direction?

- A poly-size circuit family:
  - $C_n = (x_1 \vee \neg x_1)$ if M_n halts
  - $C_n = (x_1 \wedge \neg x_1)$ if M_n loops

- decides (unary version of) HALT!
- oops…

Uniformity

- Strange aspect of circuit family:
  - can "encode" (potentially uncomputable) information in family specification
- solution: uniformity – require specification is simple to compute

Definition: circuit family $\{C_n\}$ is **logspace uniform** iff TM M outputs C_n on input 1^n and runs in O(log n) space

Uniformity

Theorem: **P** = languages decidable by logspace uniform, polynomial-size circuit families $\{C_n\}$.

- Proof:
  - already saw (⇒)
  - (⇐) on input x, generate $C_{|x|}$, evaluate it and accept iff output = 1