CS151
Complexity Theory

Lecture 3

April 9, 2019

Padding and succinctness

Two consequences of measuring running time as function of input length:

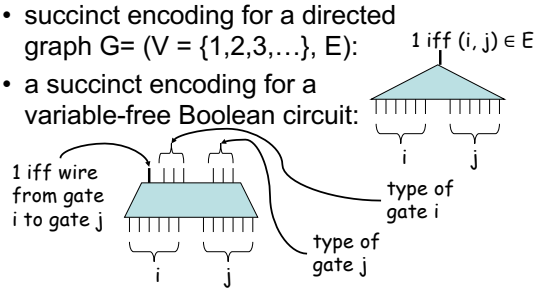
- "padding"
  - suppose L ∈ **EXP**, and define
    $PAD_L = \{ x\#^N : x \in L, N = 2^{|x|^k} \}$
  - TM that decides PAD_L: ensure suffix of N #s, ignore #s, then simulate TM that decides L
  - running time now polynomial !

Padding and succinctness

- converse (intuition only): "succinctness"
  - suppose L is **P-complete**
  - intuitively, some inputs are "hard" -- require full power of **P**
  - $SUCCINCT_L$ has inputs encoded in different form than L, some exponentially shorter
  - if "hard" inputs are exponentially shorter, then candidate to be **EXP**-complete

Succinct encodings

- succinct encoding for a directed graph G= (V = {1,2,3,…}, E):
- a succinct encoding for a variable-free Boolean circuit:



An **EXP**-complete problem

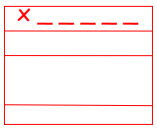
- Succinct Circuit Value: given a **succinctly encoded** variable-free Boolean circuit (gates ∧, ∨, ¬, 0, 1), does it output 1?

Theorem: Succinct Circuit Value is **EXP**-complete.

- Proof:
  - in **EXP** (why?)
  - L arbitrary language in **EXP,** TM M decides L in 2^{n^k} steps

An **EXP**-complete problem

- **tableau** for input $x = x_1x_2x_3\ldots x_n$:

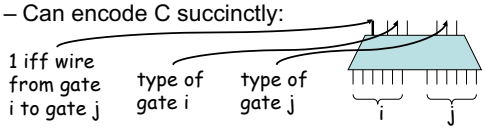


height, width 2^{n^k}

- Circuit C from CVAL reduction has size $O(2^{2n^k})$.
- TM M accepts input x iff circuit outputs 1

An **EXP**-complete problem

– Can encode C succinctly:



- if i, j within single STEP circuit, easy to compute output
- if i, j between two STEP circuits, easy to compute output
- if one of i, j refers to input gates, consult x to compute output

Summary

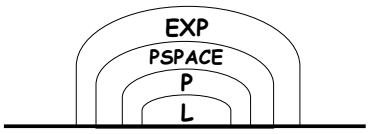
- Remaining TM details: big-oh necessary.
- First complexity classes:

  L, P, PSPACE, EXP
- First separations (via simulation and diagonalization):

  $P \neq EXP$, $L \neq PSPACE$
- First major open questions:

  $L \stackrel{?}{=} P$ $P \stackrel{?}{=} PSPACE$
- First complete problems:
  - CVAL is **P**-complete
  - Succinct CVAL is **EXP**-complete

Summary



Nondeterminism: introduction

A motivating question:

- Can computers replace mathematicians?

$L = \{ (x, 1^k) :$ statement x has a proof of length at most k $\}$

Nondeterminism: introduction

- Outline:
  - nondeterminism
  - nondeterministic time classes
  - **NP**, **NP**-completeness, **P** vs. **NP**
  - **coNP**
  - NTIME Hierarchy
  - Ladner's Theorem

Nondeterminism

- Recall deterministic TM
  - **Q** finite set of states
  - $\sum$ alphabet including blank: "_"
  - $\mathbf{q_{start}}$, $\mathbf{q_{accept}}$, $\mathbf{q_{reject}}$ in Q
  - transition function:

    $\boldsymbol{\delta} : Q \times \sum \rightarrow Q \times \sum \times \{L, R, -\}$

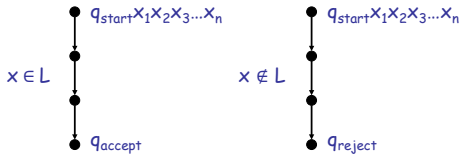
Nondeterminism

- **nondeterministic** Turing Machine:
  - **Q** finite set of states
  - ∑ alphabet including blank: "_"
  - $\mathbf{q_{start}}$, $\mathbf{q_{accept}}$, $\mathbf{q_{reject}}$ in Q
  - transition **relation**

    $\Delta \subseteq (Q \times \sum) \times (Q \times \sum \times \{L, R, -\})$
- given current state and symbol scanned, several choices of what to do next.

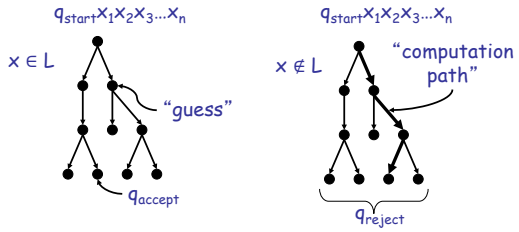
Nondeterminism

- deterministic TM: given current configuration, unique next configuration



- nondeterministic TM: given current configuration, several possible next configurations

Nondeterminism



- asymmetric accept/reject criterion

Nondeterminism

- all paths terminate
- time used: maximum length of paths from root
- space used: maximum # of work tape squares touched on any path from root

Nondeterminism

- **NTIME(f(n))** = languages decidable by a multi-tape NTM that runs for at most f(n) steps *on any computation path*, where n is the input length, and f :**N** → **N**
- **NSPACE(f(n))** = languages decidable by a multi-tape NTM that touches at most f(n) squares of its work tapes *along any computation path*, where n is the input length, and f :**N** → **N**

Nondeterminism

- Focus on time classes first:

$$\mathbf{NP} = \cup_k \mathbf{NTIME}(n^k)$$

$$\mathbf{NEXP} = \cup_k \mathbf{NTIME}(2^{n^k})$$

Poly-time verifiers

Very useful alternat "witness" or "certificate" f NP:

Theorem: language L is in NP if and only if it is expressible as: efficiently verifiable

$L = \{ x \mid \exists y, |y| \le |x|^k, (x, y) \in R \}$

where R is a language in P.

- poly-time TM M_R deciding R is a "verifier"

Poly-time verifiers

- Example: 3SAT expressible as

3SAT = {φ : φ is a 3-CNF formula for which ∃ assignment A for which (φ, A) ∈ R}

R = {(φ, A) : A is a sat. assign. for φ}

– satisfying assignmdnt A is a "witness" of the satisfiability of φ ("certifies" satisfiability of φ)
– R is decidable in poly-time

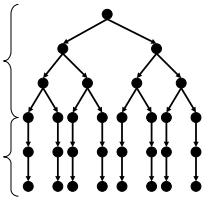
Poly-time verifiers

$L = \{ x \mid \exists y, |y| \le |x|^k, (x, y) \in R \}$

Proof: (⇒) give poly-time NTM deciding L

phase 1: "guess" y with $|x|^k$ nondeterministic steps

phase 2: decide if $(x, y) \in R$



Poly-time verifiers

Proof: (⇐) given L ∈ NP, describe L as:

$L = \{ x \mid \exists y, |y| \le |x|^k, (x, y) \in R \}$

– L is decided by NTM M running in time n^k
– define the language

R = { (x, y) : y is an accepting computation history of M on input x}

– check: accepting history has length ≤ $|x|^k$
– check: R is decidable in polynomial time
– check: M accepts x iff $\exists y, |y| \le |x|^k, (x, y) \in R$

Why NP?

problem requirements stic model of object we are seeking n

- but, captures important computational feature of many problems:

exhaustive search works

- contains **huge** number of nat problems efficient test: does y meet requirements?
- many problems have form:

$L = \{ x \mid \exists y \text{ s.t. } (x,y) \in R \}$

Why NP?

- Why not **EXP**?

– too strong!
– important problems not complete.

Relationships between classes

- Easy: **P** ⊆ **NP**, **EXP** ⊆ **NEXP**
  - TM special case of NTM
- Recall: L ∈ **NP** iff expressible as
  $L = \{ x \mid \exists y, |y| \leq |x|^k \text{ s.t. } (x,y) \in R \}$
- **NP** ⊆ **PSPACE** (try all possible y)
- The central question:

$$\mathbf{P} \overset{?}{=} \mathbf{NP}$$

finding a solution vs. recognizing a solution

NP-completeness

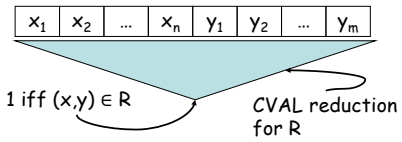
- Circuit SAT: given a Boolean circuit (gates ∧, ∨, ¬), with variables $y_1, y_2, \ldots, y_m$ is there some assignment that makes it output 1?

Theorem: Circuit SAT is **NP**-complete.

- Proof:
  - clearly in **NP**

NP-completeness

– Given L ∈ **NP** of form
$L = \{ x \mid \exists y \text{ s.t. } (x,y) \in R \}$



– hardwire input x; leave y as variables

NEXP-completeness

- Succinct Circuit SAT: given a **succinctly encoded** Boolean circuit (gates ∧, ∨, ¬), with variables $y_1, y_2, \ldots, y_m$ is there some assignment that makes it output 1?

Theorem: Succinct Circuit SAT is **NEXP**-complete.

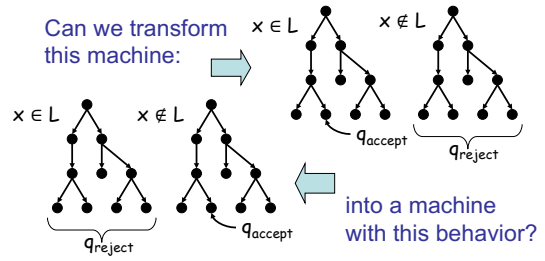
- Proof:
  - same trick as for Succinct CVAL **EXP**-complete.

Complement classes

- In general, if **C** is a complexity class
- **co-C** is the complement class, containing all complements of languages in C
  - L ∈ **C** implies (Σ* - L) ∈ **co-C**
  - (Σ* - L) ∈ **C** implies L ∈ **co-C**
- Some classes closed under complement:
  - e.g. co-P = P

coNP

- Is NP closed under complement?



coNP

- "proof system" interpretation:
- Recall: L ∈ **NP** iff expressible as

$$L = \{ x \mid \exists y, |y| \le |x|^k, (x, y) \in R \}$$

"proof" "proof verifier"

- languages in **NP** have "short proofs"
- **coNP** captures (in its complete problems) problems least likely to have "short proofs".
  - e.g., UNSAT is **coNP**-complete

coNP

- **P** = **NP** implies **NP** = **coNP**
- Belief:

NP ≠ coNP

  - another major open problem

NTIME Hierarchy Theorem

Theorem (Nondeterministic Time Hierarchy Theorem):

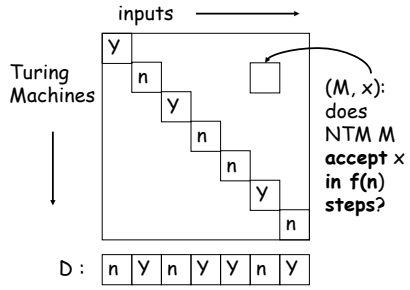
For every *proper complexity function* f(n) ≥ n, and g(n) = ω(f(n+1)),

NTIME(f(n)) ⊊ NTIME(g(n)).

NTIME Hierarchy Theorem

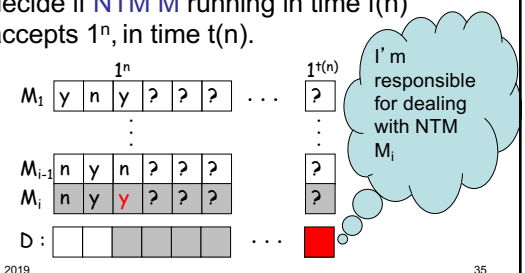
Proof attempt :

(what's wrong?)



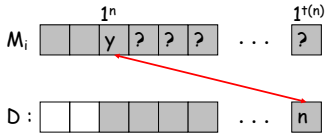
NTIME Hierarchy Theorem

- Let t(n) be large enough so that can decide if NTM M running in time f(n) accepts 1^n, in time t(n).



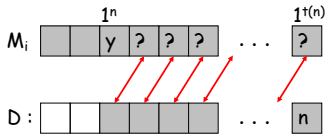
NTIME Hierarchy Theorem

- Enough time on input $1^{t(n)}$ to do the *opposite* of $M_i(1^n)$:



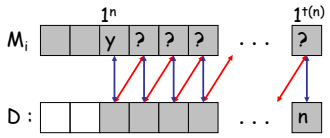
NTIME Hierarchy Theorem

- For k in [n…t(n)] can to do *same* as $M_i(1^{k+1})$ on input 1^k



NTIME Hierarchy Theorem

- Did we diagonalize against M_i?
  - if $L(M_i) = L(D)$ then:



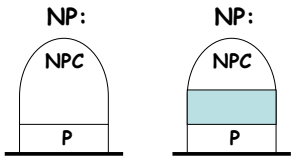
  - equality along all arrows.
  - contradiction.

NTIME Hierarchy Theorem

- General scheme:
  - interval [1...t(1)] kills M_1
  - interval [t(1)…t(t(1))] kills M_2
  - interval $[t^{i-1}(1)…t^i(1)]$ kills M_i
- Running time of D on 1^n: f(n+1) + time to compute interval containing n
- conclude D in **NTIME(g(n))** (g(n) = ω(f(n+1)))

Ladner's Theorem

- Assuming **P** ≠ **NP**, what does the world (inside **NP**) look like?



Ladner's Theorem

Theorem (Ladner): If **P** ≠ **NP**, then there exists L ∈ **NP** that is neither in **P** nor **NP**-complete.

- Proof: "lazy diagonalization"
  - deal with similar problem as in NTIME Hierarchy proof

Ladner's Theorem

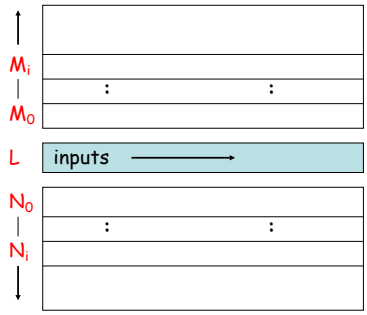
- Can enumerate (TMs deciding) all languages in **P**.
  - enumerate TMs so that each machine appears infinitely often
  - add clock to M_i so that it runs in at most n^i steps

Ladner's Theorem

- Can enumerate (TMs deciding) all **NP**-complete languages.
  - enumerate TMs f_i computing all polynomial-time functions
  - machine N_i decides language SAT reduces to via f_i *if f_i is reduction*, else SAT (details omitted…)

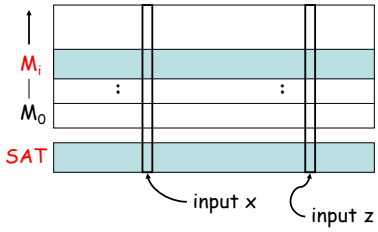
Ladner's Theorem

- Our goal: L ∈ **NP** that is neither in **P** nor **NP**-complete



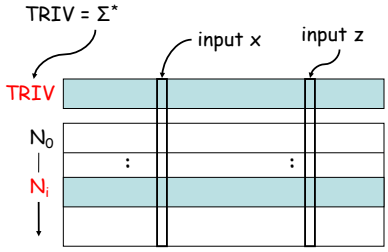
Ladner's Theorem

- Top half, assuming **P ≠ NP:**
  - focus on M_i
  - for any x, can always find some z ≥ x on which M_i and SAT differ (why?)



Ladner's Theorem

- Bottom half, assuming **P ≠ NP:**
  - focus on N_i
  - for any x, can always find some z ≥ x on which N_i and TRIV differ (why?)

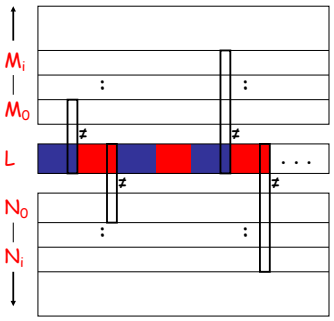


Ladner's Theorem

- Try to "merge":

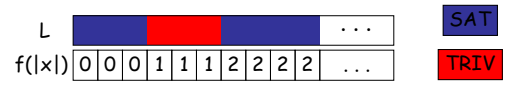
SAT TRIV

- on input x, either
  - answer SAT(x)
  - answer TRIV(x)
- if can decide which one in **P**, L ∈ **NP**



Ladner's Theorem

- General scheme: f(n) slowly increasing function



  - f(|x|) even: answer SAT(x)
  - f(|x|) odd: answer TRIV(x)
- notice choice only depends on length of input… that's OK

Ladner's Theorem

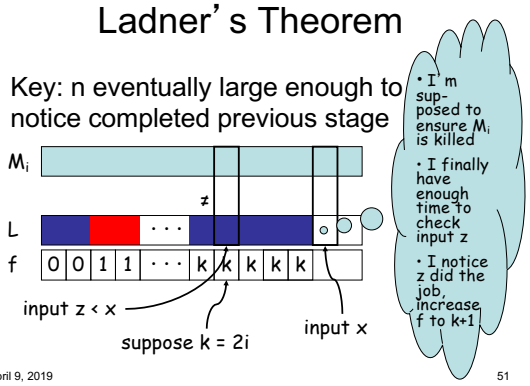
- 1st attempt to define f(n)
- "eager f(n)": increase at 1st opportunity
- Inductive definition: f(0) = 0; f(n) =
  - if f(n-1) = 2i, trying to kill M_i
    - if ∃ z < 1^n s.t. $M_i(z) \neq SAT(z)$, then f(n) = f(n-1) + 1; else f(n) = f(n-1)
  - if f(n-1) = 2i+1, trying to kill N_i
    - if ∃ z < 1^n s.t. $N_i(z) \neq TRIV(z)$, then f(n) = f(n-1) + 1; else f(n) = f(n-1)

Ladner's Theorem

- Problem: eager f(n) too difficult to compute
- on input of length n,
  - look at all strings z of length < n
  - compute SAT(z) or $N_i(z)$ for each !
- Solution: "lazy" f(n)
  - on input of length n, only run for 2n steps
  - if enough time to see should increase (over f(n-1)), do it; else, stay same
  - (alternate proof: give explicit f(n) that grows slowly enough…)

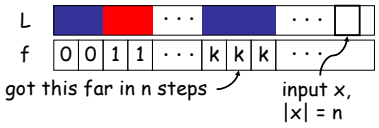
Ladner's Theorem

- Key: n eventually large enough to notice completed previous stage



Ladner's Theorem

- Inductive definition of f(n)
  - f(0) = 0
  - f(n): for n steps compute f(0), f(1), f(2),…



Ladner's Theorem

- if k = 2i:
  - for n steps try (lex order) to find z s.t. $SAT(z) \neq M_i(z)$ and f(|z|) even
  - if found, f(n) = f(n-1)+1 else f(n-1)
- if k = 2i + 1:
  - for n steps try (lex order) to find z s.t. $TRIV(z) \neq N_i(z)$ and f(|z|) odd
  - if found, f(n) = f(n-1)+1 else f(n-1)

Ladner's Theorem

- Finishing up:

  L = { x | x ∈ SAT if f(|x|) even,
  x ∈ TRIV if f(|x|) odd }

- L ∈ **NP** since f(|x|) can be computed in O(n) time

Ladner's Theorem

- suppose M_i decides L
  - f gets stuck at 2i
  - L ≡ SAT for z : |z| > n_o
  - implies SAT ∈ **P**. Contradiction.
- suppose N_i decides L
  - f gets stuck at 2i+1
  - L ≡ TRIV for z : |z| > n_o
  - implies L(N_i) ∈ **P**. Contradiction.
- (last of diagonalization…)