# CS151
# Complexity Theory

Lecture 18

May 27, 2021

## Approaches to open problems

- Almost all major open problems we have seen entail proving **lower bounds**
  - **$P \neq NP$**
  - **$L \neq P$**
  - **$P \neq PSPACE$**
  - **NC** proper
  - **$BPP \neq EXP$**
  - **PH** proper
  - **$EXP \subsetneq P/poly$**
  - **$P = BPP$ ***
  - **$NP = AM$ ***

• we know circuit lower bounds imply derandomization

• more difficult (and recent): derandomization implies circuit lower bounds!



## Approaches to open problems

- two natural approaches
  - simulation + diagonalization (uniform)
  - circuit lower bounds (non-uniform)
- no success for either approach as applied to date

Why?



## Approaches to open problems

in a precise, formal sense these approaches are **too powerful !**

- if they could be used to resolve major open problems, a side effect would be:
  - proving something that is false, or
  - proving something that is believed to be false



## Relativization

- Many proofs and techniques we have seen **relativize**:
  - they hold after replacing all TMs with oracle TMs that have access to an oracle A
  - e.g. $L^A \subseteq P^A$ for all oracles A
  - e.g. $P^A \neq EXP^A$ for all oracles A



## Relativization

- Idea: design an oracle A relative to which some statement is *false*
  - implies there can be no **relativizing proof** of that statement
  - e.g. design A for which $P^A = NP^A$
- Better: also design an oracle B relative to which statement is *true*
  - e.g. also design B for which $P^B \neq NP^B$
  - implies no **relativizing proof** can resolve truth of the statement either way !

# Relativization

- Oracles are known that falsify almost every major conjecture concerning complexity classes
  - for these conjectures, non-relativizing proofs are required
  - almost all known proofs in Complexity relativize (sometimes after some reformulation)
  - notable exceptions:
    - The PCP Theorem
    - IP = PSPACE
    - most circuit lower bounds (more on these later)



# Oracles for **P** vs. **NP**

- Goal:
  - oracle A for which $\mathbf{P}^A = \mathbf{NP}^A$
  - oracle B for which $\mathbf{P}^B \neq \mathbf{NP}^B$
- conclusion: resolving

  P vs. NP

  requires a non-relativizing proof



# Oracles for **P** vs. **NP**

- for $\mathbf{P}^A = \mathbf{NP}^A$ need A to be powerful
  - warning: intend to make **P** more powerful, but also make **NP** more powerful.
  - e.g. A = SAT doesn't work
  - however A = QSAT works:

  $$\mathbf{PSPACE} \subseteq \mathbf{P}^{QSAT} \subseteq \mathbf{NP}^{QSAT} \subseteq \mathbf{NPSPACE}$$

  and we know **NPSPACE** = **PSPACE**



# Oracles for **P** vs. **NP**

**Theorem**: there exists an oracle B for which $\mathbf{P}^B \neq \mathbf{NP}^B$.

- Proof:
  - define

  $$L = \{1^k : \exists\, x \in B \text{ s.t. } |x| = k\}$$

  - we will show $L \in \mathbf{NP}^B - \mathbf{P}^B$.
  - easy: $L \in \mathbf{NP}^B$ (no matter what B is)



# Oracles for **P** vs. **NP**

- design B by diagonalizing against all "$\mathbf{P}^B$ machines"
- $M_1, M_2, M_3, \ldots$ is an enumeration of deterministic OTMs
- each machine appears infinitely often
- $B_i$ will be those strings of length $\leq i$ in B
- we build $B_i$ after simulating machine $M_i$



# Oracles for **P** vs. **NP**

$$L = \{1^k : \exists\, x \in B \text{ s.t. } |x| = k\}$$

- Proof (continued):
  - maintain "exceptions" X that must not go in B
  - initially X = { }, $B_0$ = { }

  Stage i:
  - simulate $M_i(1^i)$ for $i^{\log i}$ steps
  - when $M_i$ makes an oracle query q:
    - if $|q| < i$, answer using $B_{i-1}$
    - if $|q| \geq i$, answer "no"; add q to X
  - if simulated $M_i$ accepts $1^i$ then $B_i = B_{i-1}$
  - if simulated $M_i$ rejects $1^i$, $B_i = B_{i-1} \cup \{x \in \{0,1\}^i : x \notin X\}$

## Oracles for **P** vs. **NP**

$L = \{1^k : \exists x \in B \text{ s.t. } |x| = k\}$

- Proof (continued):
  - if $M_i$ accepts, we ensure no strings of length i in B
  - therefore $1^i \notin L$, and so $M_i$ does not decide L
  - if $M_i$ rejects, we ensure *some* string of length i in B
  - Why?

    $B_i = B_{i-1} \cup \{x \in \{0,1\}^i : x \notin X\}$

    and |X| is at most $\Sigma_{j \le i} j^{\log j} << 2^i$
  - therefore $1^i \in L$, and so $M_i$ does not decide L
  - Conclude: $L \notin \mathbf{P}^{\mathbf{B}}$



## Circuit lower bounds

- Relativizing techniques are out…
- but most circuit lower bound techniques do not relativize
- exponential circuit lower bounds known for weak models:
  - e.g. constant-depth poly-size circuits
- But, utter failure (so far) for more general models. Why?



## Natural Proofs

- Razborov and Rudich defined the following "natural" format for circuit lower bounds:
  - identify property **P** of functions $f:\{0,1\}^* \rightarrow \{0,1\}$
  - $\mathbf{P} = \cup_n \mathbf{P}_n$ is a natural property if:
    - (useful) $\forall n$ $f_n \in \mathbf{P}_n$ implies f does not have poly-size circuits [i.e. $f_n \in \mathbf{P}_n$ implies ckt size $\ge s(n) >> poly(n)$]
    - (constructive) can decide "$f_n \in \mathbf{P}_n$?" in poly time given the *truth table* of $f_n$
    - (large) at least $(½)^{O(n)}$ fraction of all $2^{2^n}$ functions on n bits are in $\mathbf{P}_n$
  - show some function family $g = \{g_n\}$ is in $\mathbf{P}_n$



## Natural Proofs

- *all known circuit lower bound techniques are natural* for a suitably parameterized version of the definition

**Theorem** (RR): if there is a $2^{n^\epsilon}$-OWF, then there is no natural property **P**.
  - factoring believed to be $2^{n^\epsilon}$-OWF
  - general version also rules out natural properties useful for proving many other separations, under similar cryptographic assumptions



## Natural Proofs

- Proof:
  - **main idea**: natural property $\mathbf{P}_n$ can efficiently distinguish

    pseudorandom functions

    from

    truly random functions
  - but cryptographic assumption implies existence of *pseudorandom functions* for which this is impossible



## Proof (continued)

- Recall: assuming One-Way-Permutations

  $f_k:\{0,1\}^k \rightarrow \{0,1\}^k$

  that are not invertible by $2^{k^\epsilon}$ size circuits
- we constructed PRG $G:\{0,1\}^k \rightarrow \{0,1\}^{2k}$
  - no circuit C of size $s = 2^{k^\delta}$ for which

    $|Pr_x[C(G(x)) = 1] - Pr_z[C(z) = 1]| > 1/s$

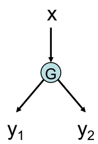
(BMY construction with slightly modified parameters)

# Proof (continued)

- Think of G as $G:\{0,1\}^k \rightarrow \{0,1\}^k \times \{0,1\}^k$

  $G(x) = (y_1, y_2)$

- Graphically:



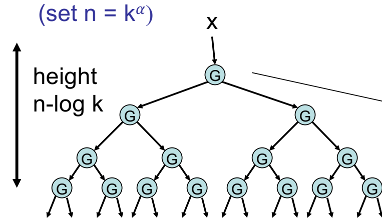



# Proof (continued)

- A function $F:\{0,1\}^k \rightarrow \{0,1\}^{2^n}$

  (set $n = k^\alpha$)

Given x, i, can compute i-th output bit in time n·poly(k)




each x, defines a poly-time computable function $f_x$



# Proof (continued)

(useful) $\forall n\ f_n \in \underline{\mathbf{P}}_n \Rightarrow f$ does not have poly-size circuits
(constructive) "$f_n \in \underline{\mathbf{P}}_n$?" in poly time given *truth table* of $f_n$
(large) at least $(½)^{O(n)}$ fraction of all $2^{2^n}$ fns. on n-bits in $\underline{\mathbf{P}}_n$

- $f_x$ in poly-time $\Rightarrow$ for all x: $f_x \notin \underline{\mathbf{P}}_n$ (useful)
- $\Pr_g[g \in \underline{\mathbf{P}}_n] \geq (1/2)^{O(n)}$ (large)
- constructive: exists circuit $T:\{0,1\}^{2^n} \rightarrow \{0,1\}$ of size $2^{O(n)}$ for which

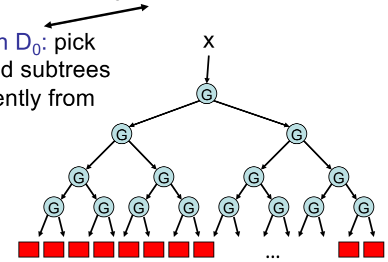
  $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$



# Proof (continued)

- $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_0$: pick roots of red subtrees independently from $\{0,1\}^k$

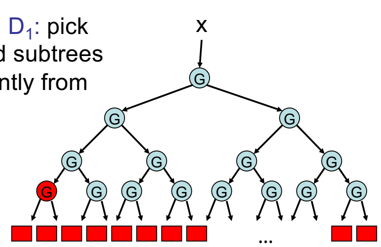





# Proof (continued)

- $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_1$: pick roots of red subtrees independently from $\{0,1\}^k$

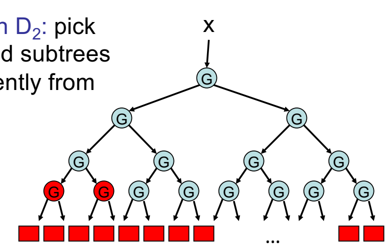





# Proof (continued)

- $|\Pr_x[T(f_x) = 1] - \Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

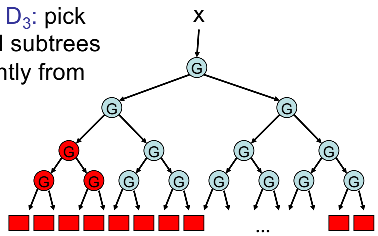
distribution $D_2$: pick roots of red subtrees independently from $\{0,1\}^k$



## Proof (continued)

- $|Pr_x[T(f_x) = 1] – Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_3$: pick roots of red subtrees independently from $\{0,1\}^k$

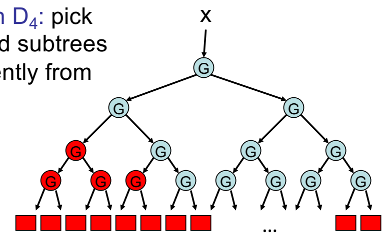





## Proof (continued)

- $|Pr_x[T(f_x) = 1] – Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_4$: pick roots of red subtrees independently from $\{0,1\}^k$

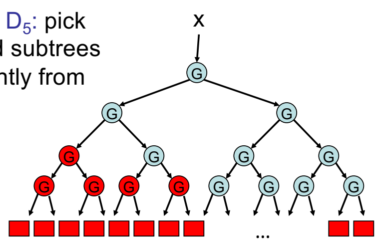





## Proof (continued)

- $|Pr_x[T(f_x) = 1] – Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_5$: pick roots of red subtrees independently from $\{0,1\}^k$

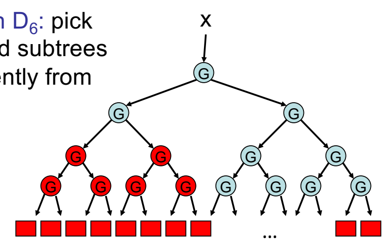





## Proof (continued)

- $|Pr_x[T(f_x) = 1] – Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_6$: pick roots of red subtrees independently from $\{0,1\}^k$

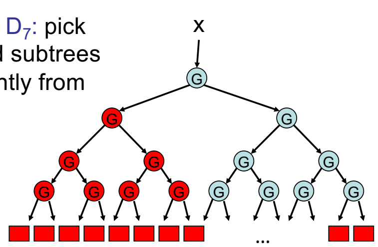





## Proof (continued)

- $|Pr_x[T(f_x) = 1] – Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

distribution $D_7$: pick roots of red subtrees independently from $\{0,1\}^k$

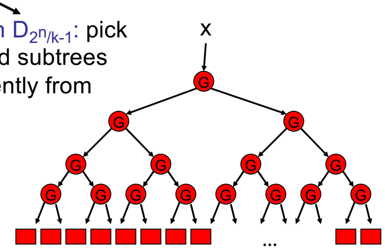





## Proof (continued)

- $|Pr_x[T(f_x) = 1] – Pr_g[T(g) = 1]| \geq (1/2)^{O(n)}$

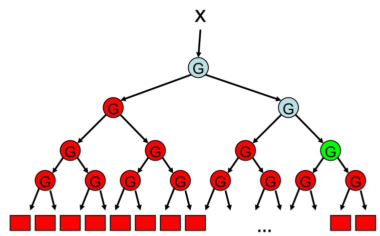
distribution $D_{2^{n/k}-1}$: pick roots of red subtrees independently from $\{0,1\}^k$



## Proof (continued)

– For some i:

$|\Pr[T(D_i) = 1] - \Pr[T(D_{i-1}) = 1]| \geq (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$



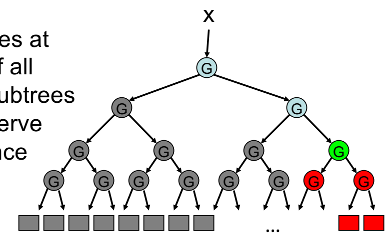



## Proof (continued)

– For some i:

$|\Pr[T(D_i) = 1] - \Pr[T(D_{i-1}) = 1]| \geq (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$

fix values at roots of all other subtrees to preserve difference



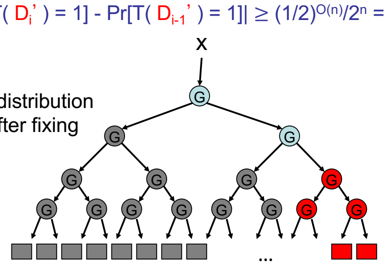



## Proof (continued)

– For some i:

$|\Pr[T( D_i' ) = 1] - \Pr[T( D_{i-1}' ) = 1]| \geq (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$

$D_i'$: distribution $D_i$ after fixing



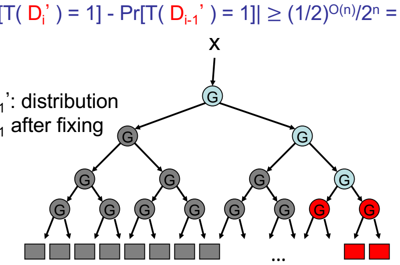



## Proof (continued)

– For some i:

$|\Pr[T( D_i' ) = 1] - \Pr[T( D_{i-1}' ) = 1]| \geq (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$

$D_{i-1}'$: distribution $D_{i-1}$ after fixing

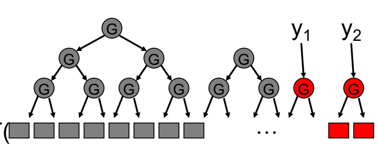





## Proof (continued)

$|\Pr[T( D_i' ) = 1] - \Pr[T( D_{i-1}' ) = 1]| \geq (1/2)^{O(n)}/2^n = (1/2)^{O(n)}$

– $C(y_1,y_2)=T($




$)$

$|\Pr_x[C(G(x)) = 1] - \Pr_{y_1, y_2}[C(y_1, y_2) = 1]| \geq (1/2)^{O(n)}$

T( $D_i'$ )     T( $D_{i-1}'$ )



## Proof (continued)

– recall: no circuit C of size $s = 2^{k^\delta}$ for which:

$|\Pr_x[C(G(x)) = 1] – \Pr_{y_1, y_2}[C(y_1, y_2) = 1]| > 1/s$

– we have C of size $2^{O(n)}$ for which:

$|\Pr_x[C(G(x)) = 1] - \Pr_{y_1, y_2}[C(y_1, y_2) = 1]| \geq (1/2)^{O(n)}$

– with $n = k^\alpha$, $\alpha$ arbitrary constant

– set $\alpha$ such that $2^{O(n)} < s$

– contradiction.

## Natural Proofs

- To prove circuit lower bounds, we must either:
  - Violate largeness: seize upon an incredibly specific feature of hard functions (one not possessed by a random function ! )
  - Violate constructivity: identify a feature of hard functions that cannot be computed efficiently from the **truth table**
- no "non-natural property" known for all but the very weakest models…



"We do not conclude that researchers should give up on proving serious lower bounds.



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"We do not conclude that researchers should give up on proving serious lower bounds. Quite the contrary, by classifying a large number of techniques that are unable to do the job, we hope to focus research in a more fruitful direction. Pessimism will only be warranted if a *long period of time* passes without the discovery of a non-naturalizing lower bound proof."

**Rudich and Razborov**
**1994**



## Moral

- To resolve central questions:
  - avoid relativizing arguments
    - use PCP theorem and related results
    - focus on circuits, etc…
  - avoid constructive arguments
  - avoid arguments that yield lower bounds for random functions



## Course Summary

## Course summary

- <u>Time and space</u>
  - hierarchy theorems
  - FVAL in **L**
  - CVAL **P**-complete
  - QSAT **PSPACE**-complete
  - succinct CVAL **EXP**-complete



## Course summary

- <u>Non-determinism</u>
  - NTIME hierarchy theorem
  - "NP-intermediate" problems (Ladner's Theorem)
  - unary languages (likely) not NP-complete
  - Savitch's Theorem
  - Immerman-Szelepcsényi Theorem

  Problem sets:
  - *sparse* languages (likely) not NP-complete



## Course summary

- <u>Non-uniformity</u>
  - formula lower bound (Andreev, Hastad)
  - monotone circuit lower bound (Razborov)

  Problem sets:
  - Barrington's Theorem
  - formula lower bound for parity



## Course summary

- <u>Randomness</u>
  - polynomial identity testing + Schwartz-Zippel
  - unique-SAT (Valiant-Vazirani Theorem)
  - Blum-Micali-Yao PRG
  - Nisan-Wigderson PRG
  - worst-case hardness ) average-case hardness
  - Trevisan extractor

  Problem sets:
  - Goldreich-Levin hard bit



## Course summary

- <u>Alternation</u>
  - $\text{QSAT}_i$ complete for levels of the **PH**
  - Karp-Lipton theorem
  - **BPP** in **PH**

  Problem sets:
  - approximate counting + sampling with an **NP**-oracle
  - **VC**-dimension is $\Sigma_3$-complete
  - the class $S_2^P$ (final)



## Course summary

- <u>Counting</u>
  - #matching is **#P**-complete

  Problem sets:
  - permanent is **#P**-complete
  - Toda's theorem: $\mathbf{PH} \subseteq \mathbf{P}^{\#\mathbf{P}}$

## Course summary

- Interaction
  - **IP** = **PSPACE**
  - GI in **NP** ∩ **coAM**
  - using NW PRG for **MA**, variant for **AM**
  - hardness of approximation , PCPs
  - elements of the PCP theorem

  Problem sets:
  - BLR linearity test
  - Clique hard to approximate to within $N^{\epsilon}$



## Course summary

- Barriers to progress
  - oracles rule out relativizing proofs
  - "natural proofs" rule out many circuit lower bound techniques



## Course summary

| | |
|---|---|
| • Time and space | **L, P, PSPACE, EXP** |
| • Non-determinism | **NL, NP, coNP, NEXP** |
| • Non-uniformity | **NC, P/poly** |
| • Randomness | **RL, ZPP, RP, coRP, BPP** |
| • Alternation | **PH, PSPACE** |
| • Counting | **#P** |
| • Interaction | **IP, MA, AM, PCP[log n, 1]** |



## The big picture

- All classes on previous slide are probably distinct, except:
  - **P, ZPP, RP, coRP, BPP** (probably all equal)
  - **L, RL** (probably all equal; **NL**?)
  - **NP, MA, AM** (probably all equal)
  - **IP = PSPACE**
  - **PCP[log n, 1] = NP**
- Only real separations we know separate classes delimiting same resource:
  - e.g. **L** ≠ **PSPACE, NP** ≠ **NEXP**



## The big picture

Remember:

possible explanation for failure to prove conjectured separations…

…is that they are false



## The big picture

- Important techniques/ideas:
  - simulation and diagonalization
  - reductions and completeness
  - self-reducibility
  - encoding information using low-degree polynomials
  - randomness
  - others…

## The big picture

- I hope you take away:
  - an ability to extract the essential features of a problem that make it hard/easy…
  - knowledge and tools to connect computational problems you encounter with larger questions in complexity
  - background needed to understand current research in this area



## The big picture

- background to *contribute* to current research in this area
  - many open problems
  - young field
  - try your hand…