CS151 Complexity Theory

Lecture 16
May 25, 2023

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MAX-k-SAT

- Missing link: first gap-producing reduction
  - history's guide

    it should have something to do with SAT
- Definition: MAX-k-SAT with gap ε
  - instance: k-CNF φ
  - YES: some assignment satisfies **all** clauses
  - NO: no assignment satisfies more than (1 – ε) fraction of clauses

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Proof systems viewpoint

- MAX-k-SAT with gap ε **NP**-hard ⇒ for any language L ∈ **NP** proof system of form:
  - given x, compute reduction to MAX-k-SAT: ϕ_x
  - expected proof is satisfying assignment for ϕ_x
  - verifier picks random clause ("local test") and checks that it is satisfied by the assignment

    x ∈ L ⇒ Pr[verifier accepts] = 1

    x ∉ L ⇒ Pr[verifier accepts] ≤ (1 – ε)
  - can repeat O(1/ε) times for error < ½

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Proof systems viewpoint

- can think of reduction showing k-SAT NP-hard as designing a proof system for **NP** in which:
  - verifier only performs local tests
- can think of reduction showing "MAX-k-SAT with gap ε" NP-hard as designing a proof system for **NP** in which:
  - verifier only performs local tests
  - invalidity of proof* evident all over: "holographic proof" and an ϵ fraction of tests notice such invalidity

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PCP

- Probabilistically Checkable Proof (PCP) permits novel way of verifying proof:
  - pick random local test
  - query proof in specified k locations
  - accept iff passes test
- fancy name for a NP-hardness reduction

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PCP

- **PCP[r(n),q(n)]**: set of languages L with p.p.t. verifier V that has (r, q)-restricted access to a string "proof"
  - V tosses O(r(n)) coins
  - V accesses proof in O(q(n)) locations
  - (completeness) x ∈ L ⇒ ∃ proof such that

    Pr[V(x, proof) accepts] = 1
  - (soundness) x ∉ L ⇒ ∀ proof*

    Pr[V(x, proof*) accepts] ≤ ½

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PCP

- Two observations:
  - **PCP[1, poly n] = NP**
    proof?
  - **PCP[log n, 1] ⊆ NP**
    proof?

The PCP Theorem (AS, ALMSS):

PCP[log n, 1] = NP.

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PCP

Corollary: MAX-k-SAT is **NP**-hard to approximate to within some constant ϵ.

- using PCP[log n, 1] protocol for, say, VC
- enumerate all $2^{O(\log n)} = \text{poly}(n)$ sets of queries
- construct a k-CNF φ_i for verifier's test on each
  - note: k-CNF since function on only k bits
- "YES" VC instance ⇒ all clauses satisfiable
- "NO" VC instance ⇒ every assignment fails to satisfy at least ½ of the φ_i ⇒ fails to satisfy an $\epsilon = (½)2^{-k}$ fraction of clauses.

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The PCP Theorem

- Elements of proof:
  - arithmetization of 3-SAT
    - we will do this
  - low-degree test
    - we will state but not prove this
  - self-correction of low-degree polynomials
    - we will state but not prove this
  - proof composition
    - we will describe the idea

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The PCP Theorem

- Two major components:
  - **NP ⊆ PCP[log n, polylog n]** ("outer verifier")
    - we will prove this from scratch, assuming low-degree test, and self-correction of low-degree polynomials
  - **NP ⊆ PCP[n², 1]** ("inner verifier")
    - we will prove (low-degree test on Problem Set)

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Proof Composition (idea)

NP ⊆ PCP[log n, polylog n] ("outer verifier")
NP ⊆ PCP[n², 1] ("inner verifier")

- **composition** of verifiers:
  - reformulate "outer" so that it uses O(log n) random bits to make **1** query to each of **3** provers
  - replies r_1, r_2, r_3 have length polylog n
  - Key: accept/reject decision computable from r_1, r_2, r_3 by small circuit C

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Proof Composition (idea)

NP ⊆ PCP[log n, polylog n] ("outer verifier")
NP ⊆ PCP[n², 1] ("inner verifier")

- **composition** of verifiers (continued):
  - final proof contains **proof** that $C(r_1, r_2, r_3) = 1$ for inner verifier's use
  - use inner verifier to verify that $C(r_1,r_2,r_3) = 1$
  - O(log n)+polylog n randomness
  - O(1) queries
  - tricky issue: consistency

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Proof Composition (idea)

- $NP \subseteq PCP[\log n, 1]$ comes from
  - repeated composition
  - $PCP[\log n, \text{polylog } n]$ with $PCP[\log n, \text{polylog } n]$ yields $PCP[\log n, \text{polyloglog } n]$
  - $PCP[\log n, \text{polyloglog } n]$ with $PCP[n^2, 1]$ yields $PCP[\log n, 1]$
- details omitted…

The inner verifier

Theorem: **$NP \subseteq PCP[n^2, 1]$**

Proof (first steps):

1. Quadratic Equations is NP-hard
2. PCP for QE:

   proof = *all quadratic functions* of a soln. x

   verification = check that a random linear combination of equations is satisfied by x

(if prover keeps promise to supply all quadratic fns of x)

Quadratic Equations

- quadratic equation over F_2:

  $$\sum_{i<j} a_{i,j} X_i X_j + \sum_i b_i X_i + c = 0$$

- language QUADRATIC EQUATIONS (QE)

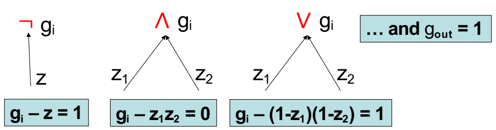
  = { systems of quadratic equations over F_2 that have a solution (assignment to the X variables) }

Quadratic Equations

Lemma: QE is NP-complete.

Proof: clearly in NP; reduce from CIRCUIT SAT

- circuit C an instance of CIRCUIT SAT
- QE variables = variables + gate variables



Quadratic Functions Code

- intended proof:
  - F the field with 2 elements
  - given $x \in F^n$, a solution to instance of QE
  - $f_x: F^n \rightarrow F_2$ all linear functions of x

    $$f_x(a) = \sum_i a_i x_i$$

  - $g_x: F^{n \times n} \rightarrow F_2$ **includes** all quadratic fns of x

    $$g_x(A) = \sum_{i,j} A[i,j]x_i x_j$$

  - KEY: can evaluate any quadratic function of x with a single evaluation of f_x and g_x

PCP for QE

If prover keeps promise to supply all quadratic fns of x, a solution of QE instance…

- Verifier's action:

–query a *random linear combination* R of the equations of the QE instance

–<u>Completeness</u>: obvious

–<u>Soundness</u>: x fails to satisfy some equation; imagine picking coeff. for this one last

$$\Pr[x \text{ satisfies } R] = 1/2$$

PCP for QE

$x \in F^n$ soln	$f_x(a) = \sum_i a_i x_i$	Had(x)
	$g_x(A) = \sum_{i,j} A[i,j]x_i x_j$	Had($x \otimes x$)

To "enforce promise", verifier needs to perform:

- linearity test: verify f, g are (close to) linear
- self-correction: access the *linear* f', g' that are close to f, g
  [so f' = Had(u) and g' = Had(V)]
- consistency check: verify $V = u \otimes u$

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PCP for QE

$x \in F^n$ soln	$f_x(a) = \sum_i a_i x_i$	Had(x)
	$g_x(A) = \sum_{i,j} A[i,j]x_i x_j$	Had($x \otimes x$)

- Linearity test: given access to $h:F^m \rightarrow F$
  - pick random a,b; check if h(a) + h(b) = h(a+b); repeat O(1) times
  - do this for functions f and g supplied by prover

Theorem [BLR]: h linear ⇒ prob. success = 1; prob. success $\geq 1 - \delta$ ⇒ ∃ linear h' s.t.

$$\Pr_a[h'(a) = h(a)] \geq 1 - O(\delta)$$

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PCP for QE

$x \in F^n$ soln	$f_x(a) = \sum_i a_i x_i$	Had(x)
	$g_x(A) = \sum_{i,j} A[i,j]x_i x_j$	Had($x \otimes x$)

- Self-correction:
  - given access to $h:F^m \rightarrow F$ close to linear h'; i.e.,
    $$\Pr_a[h'(a) = h(a)] \geq 1 - O(\delta)$$
  - to access h'(a), pick random b; compute
    $$h(b) + h(a+b)$$
  - with prob. at least $1 - 2\cdot O(\delta)$, h(b) = h'(b) and h(a+b) = h'(a+b); hence we compute h'(a)

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PCP for QE

$x \in F^n$ soln	$f_x(a) = \sum_i a_i x_i$	Had(x)
	$g_x(A) = \sum_{i,j} A[i,j]x_i x_j$	Had($x \otimes x$)

- Consistency check: given access to linear functions f' = Had(u) and g' = Had(V)
  - pick random $a, b \in F^n$; check that
    $$f'(a)f'(b) = g'(ab^T)$$
  - completeness: if $V = u \otimes u$
    $$f'(a)f'(b) = (\sum_i a_i u_i)(\sum_i b_i u_i) = \sum_{i,j} a_i b_j V[i,j] = g'(ab^T)$$

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PCP for QE

$x \in F^n$ soln	$f_x(a) = \sum_i a_i x_i$	Had(x)
	$g_x(A) = \sum_{i,j} A[i,j]x_i x_j$	Had($x \otimes x$)

- Consistency check: given access to linear functions f' = Had(u) and g' = Had(V)
  - soundness: claim that if V ≠

$$\Pr[(\sum_i a_i u_i)(\sum_i b_i u_i) = \sum_{i,j} a$$

∃ i,j s.t. uu^T and V differ in entry (i,j);

∃ i s.t. $(uu^T)b$ and Vb differ in entry i; pick a_i last

$$\Pr[(uu^T)b \neq Vb$$

$$\Pr[a^T(uu^T)b \neq a^TVb] \geq ½\cdot½ = ¼$$

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The outer verifier

Theorem: **NP ⊆ PCP[log n, polylog n]**

Proof (first steps):

- define: Polynomial Constraint Satisfaction (PCS) problem
- prove: PCS gap problem is **NP**-hard

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NP ⊆ PCP[log n, polylog n]

- MAX-k-SAT
  - given: k-CNF ϕ
  - output: max. # of simultaneously satisfiable clauses
- generalization: MAX-k-CSP
  - given:
    - variables $x_1, x_2, \ldots, x_n$ taking values from set S
    - k-ary constraints $C_1, C_2, \ldots, C_t$
  - output: max. # of simultaneously satisfiable constraints

NP ⊆ PCP[log n, polylog n]

- algebraic version: MAX-k-PCS
  - given:
    - variables $x_1, x_2, \ldots, x_n$ taking values from **field** F_q
    - $n = q^m$ for some integer m
    - k-ary constraints $C_1, C_2, \ldots, C_t$
  - assignment viewed as $f:(F_q)^m \rightarrow F_q$
  - output: max. # of constraints simultaneously satisfiable by an assignment that has deg. ≤ d

NP ⊆ PCP[log n, polylog n]

- MAX-k-PCS **gap problem**:
  - given:
    - variables $x_1, x_2, \ldots, x_n$ taking values from **field** F_q
    - $n = q^m$ for some integer m
    - k-ary constraints $C_1, C_2, \ldots, C_t$
  - assignment viewed as $f:(F_q)^m \rightarrow F_q$
  - YES: some degree d assignment satisfies all constraints
  - NO: no degree d assignment satisfies more than $(1-\epsilon)$ fraction of constraints

NP ⊆ PCP[log n, polylog n]

Lemma: for every constant $1 > \varepsilon > 0$, the MAX-k-PCS gap problem with

t = poly(n) k-ary constraints with k = polylog(n)
field size q = polylog(n)
$n = q^m$ variables with $m = O(\log n / \log\log n)$
degree of assignments d = polylog(n)
gap ϵ

is **NP**-hard.

NP ⊆ PCP[log n, polylog n]

t = poly(n) k-ary constraints with k = polylog(n)
field size q = polylog(n)
$n = q^m$ variables with $m = O(\log n / \log\log n)$
degree of assignments d = polylog(n)

- check: headed in right direction
  - O(log n) random bits to pick a constraint
  - query assignment in O(polylog(n)) locations to determine if it is satisfied
  - completeness 1; soundness $1-\epsilon$

  (if prover keeps promise to supply degree d polynomial)

NP ⊆ PCP[log n, polylog n]

- Proof of Lemma
  - reduce from 3-SAT
  - 3-CNF $\varphi(x_1, x_2, \ldots, x_n)$
  - can encode as $\psi:[n] \times [n] \times [n] \times \{0,1\}^3 \rightarrow \{0,1\}$
  - $\psi(i_1, i_2, i_3, b_1, b_2, b_3) = 1$ iff φ contains clause $(x_{i_1}^{b1} \vee x_{i_2}^{b2} \vee x_{i_3}^{b3})$
  - e.g. $(x_3 \vee \neg x_5 \vee x_2) \Rightarrow \psi(3,5,2,1,0,1) = 1$

NP ⊆ PCP[log n, polylog n]

– pick $H \subseteq F_q$ with $\{0,1\} \subseteq H$, $|H|$ = polylog n
– pick $m = O(\log n/\log\log n)$ so $|H|^m = n$
– identify $[n]$ with H^m
– $\psi: H^m \times H^m \times H^m \times H^3 \rightarrow \{0,1\}$ encodes φ
– assignment $a: H^m \rightarrow \{0,1\}$
– Key: a satisfies φ iff $\forall i_1,i_2,i_3,b_1,b_2,b_3$
$\psi(i_1,i_2,i_3,b_1,b_2,b_3) = 0$ or
$a(i_1)=b_1$ or $a(i_2)=b_2$ or $a(i_3)=b_3$

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NP ⊆ PCP[log n, polylog n]

$\psi: H^m \times H^m \times H^m \times H^3 \rightarrow \{0,1\}$ encodes φ
a satisfies φ iff $\forall i_1,i_2,i_3,b_1,b_2,b_3$
$\psi(i_1,i_2,i_3,b_1,b_2,b_3) = 0$ or $a(i_1)=b_1$ or $a(i_2)=b_2$ or $a(i_3)=b_3$

– **extend** ψ to a function $\psi':(F_q)^{3m+3} \rightarrow F_q$ with degree at most $|H|$ in each variable
– can **extend** any assignment $a: H^m \rightarrow \{0,1\}$ to $a':(F_q)^m \rightarrow F_q$ with degree $|H|$ in each variable

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NP ⊆ PCP[log n, polylog n]

$\psi':(F_q)^{3m+3} \rightarrow F_q$ encodes φ
$a':(F_q)^m \rightarrow F_q$ s.a. iff $\forall (i_1,i_2,i_3,b_1,b_2,b_3) \in H^{3m+3}$
$\psi'(i_1,i_2,i_3,b_1,b_2,b_3) = 0$ or $a'(i_1)=b_1$ or $a'(i_2)=b_2$ or $a'(i_3)=b_3$

– define: $p_{a'}:(F_q)^{3m+3} \rightarrow F_q$ from a' as follows
$p_{a'}(i_1,i_2,i_3,b_1,b_2,b_3) =$
$\psi'(i_1,i_2,i_3,b_1,b_2,b_3)(a'(i_1) - b_1)(a'(i_2) - b_2)(a'(i_3) - b_3)$
– a' s.a. iff $\forall (i_1,i_2,i_3,b_1,b_2,b_3) \in H^{3m+3}$
$p_{a'}(i_1,i_2,i_3,b_1,b_2,b_3) = 0$

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