1. The East Model is a Markov chain with state space \( \Omega = \{ x = x_1x_2\ldots x_{n+1} \in \{0,1\}^{n+1} : x_{n+1} = 1 \} \) with transitions defined as follows:

Select a coordinate \( i \in \{1,\ldots,n\} \) uar. If \( x_{i+1} = 1 \) then flip the value at \( i \) (i.e., replace \( x_i \) by \( 1 - x_i \)), else do nothing.

(a) Show that this Markov chain is irreducible and aperiodic, and that its stationary distribution is uniform.

(b) Show that the mixing time is at least \( n^2 - cn^{3/2} \) for some constant \( c \).

(Hint: consider the time to go from the state \( \bar{0}1 \) to any state of the form \( 1x_2\ldots x_n1 \). You might need Chebyshev’s inequality and the fact that a geometric random variable with parameter \( p \) has expectation \( \frac{1}{p} \) and variance \( \frac{1-p}{p^2} \).)

2. Let \( 1 \leq k \leq n/2 \), and let \( \Omega \) be the set of all subsets of \( [n] \) of cardinality \( k \). Consider the following Markov chain on \( \Omega \): from a subset \( S \), pick an element \( a \in S \) and an element \( b \in [n] \setminus S \), independently and uar, and move to the set \( S + b - a \).

(a) Show that this Markov chain (augmented if you like by a uniform self-loop probability) is ergodic with uniform stationary distribution.

(b) Devise a coupling argument that shows that the mixing time is asymptotically \( O(k \log k) \).