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Problem set 3 **Out Friday Nov 8, Due Tuesday Nov 26 in Jenish's mailbox.**

1. (Warm-up exercise, not to be handed in): Recall the coupon collector problem: there are n coupons. In each round you sample a single coupon, uniformly (with repetition). T is the time until you have seen every kind of coupon at least once. Show that

$$\Pr(T > n \ln n + cn) \leq e^{-c}.$$

(By the way, the limit of the LHS for fixed c and large n is $1 - e^{-e^{-c}}$.)

2. In this problem we continue from ps2-3. Again, you have to choose among n actions at every discrete time $t = 1, 2,$ etc.; after each round, all possible penalties $M_k(t)$ $(1 \leq k \leq n)$ (including the one you incurred) are revealed. Penalties are in the range $[0, 1]$.

   In ps2-3 you were to show that MWU can achieve the following performance:

$$\sum_t \sum_k p_k(t-1) M_k(t) \leq 2\sqrt{T \ln n} + \min_k \sum_t M_k(t).$$

   Now I want you to show that for $T \geq 100 \ln n$, there is a matching $\Omega(\sqrt{T \ln n})$ lower bound, for any online algorithm, on the expected regret. (Remember, regret = gap between the cost of the algorithm and the (a posteriori) best choice of k.) Hint: let each $M_k(t)$ be an iid Benoulli rv, uniform in $\{0, 1\}$.

3. In the min-cut algorithm we want to store a weighted (undirected, simple) graph on n vertices, with edge-weights w, as a weighted adjacency matrix and be able to perform the following two operations:

   (a) Sample an edge $\{i, j\}$ with probability $\frac{w(i,j)}{\sum_{k<\ell} w(k,\ell)}$.

   (b) Contract an edge (i.e., produce an appropriate $(n-1) \times (n-1)$ weighted adjacency matrix). We should also maintain "pointers" that let us reconstruct the pair of row and columns that were contracted into one.

   Show how to do these things in time $O(n)$.

4. For positive integer z, a proper z-coloring of a graph $G = (V, E)$ is a mapping g from V into $[z]$ such that for every $\{u, v\} \in E$, $g(u) \neq g(v)$. Define the following function on graphs G and positive integers z:

$$\chi(G, z) = |\{g : g \text{ is a proper } z\text{-coloring of } G\}|$$

   Recall from lecture the notion of an *edge contraction* of an undirected graph. Generally (and as done in class), that notion requires keeping track of edge weights, but for the present purpose, it makes sense to think of edge contraction as a mapping that takes a simple undirected graph, and an edge $\{u, v\}$ in that graph, and produces a new simple undirected graph $G/\{u, v\}$ which is the same as G except that it is missing vertices u and v, has a new vertex $\{u, v\}$, and the new vertex has edge $\{\{u, v\}, w\}$ (for $w \notin \{u, v\}$) if either $\{u, w\}$ or $\{v, w\}$ are in E.

   We also require the notion of an *edge deletion:* this (again provided $\{u, v\} \in E$) produces the simple undirected graph $G \backslash \{u, v\}$ which has the same vertex set as G and is missing only the edge $\{u, v\}$.

(a) Show that

$$\chi(G,z) = \chi(G \backslash \{u,v\}, z) - \chi(G/\{u,v\}, z)$$

(b) Show that for any fixed G on n vertices, $\chi(G,z)$ is a polynomial of degree n in z.

(c) There are some points where it is quite easy to evaluate the chromatic polynomial. Show, specifically, that the values $\chi(G,1)$ and $\chi(G,2)$ can be evaluated in polynomial time. Show also that there cannot be $n+1$ integers z in $[n^5]$ at which there is a polynomial time (or randomized polynomial time) algorithm to evaluate $\chi(G,z)$ unless NP is inside P (or BPP).

(d) Prove that $(-1)^n \chi(G,-1)$ counts the number of *acyclic orientations* of G. (An acyclic orientation is a selection, for every edge $\{u,v\}$, of either the orientation $u \rightarrow v$ or the orientation $v \rightarrow u$, such that the resulting digraph has no directed cycles.)

5. Let A be a square matrix with nonnegative entries. We say that A is *irreducible* if $\forall i,j\ \exists k : (A^k)_{ij} > 0$. We say that A is *aperiodic* if $\forall i,j : \gcd\{k : (A^k)_{i,j} > 0\} = 1$.

   The *period* of state i is $\gcd\{k : (A^k)_{ii} > 0\}$. Show that if A is irreducible then

   (a) the period of every i is the same,

   (b) $[\exists k : \forall i,j : (A^k)_{ij} > 0] \iff A$ is aperiodic.

6. Show that if square matrix A with nonnegative entries is irreducible and aperiodic then all the conclusions of Perron's theorem still go through.