3.6 Min-cut and network reliability

3.6.1 Min-Cut: Karger's algorithm

We consider undirected simple (no loops or multiple edges) weighted graphs G on n vertices.

$$w(i,j) \geq 0, \quad w(i,j) = w(j,i), \quad i \neq j \in \{1,\ldots,n\}$$

Write $w(G) = \sum_{k<\ell} w(k,\ell)$.

A *cut* is a partition of the vertices into two non-empty subsets $S, \bar{S}$; the weight of cut $(S,\bar{S})$ is

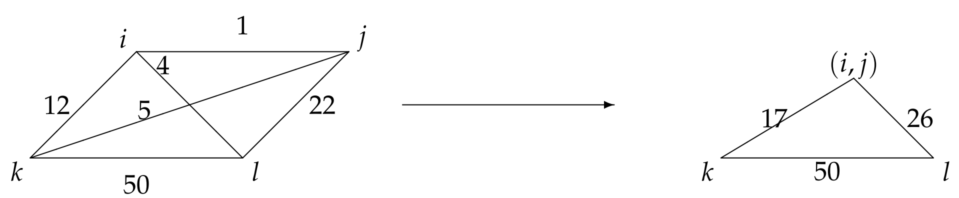
$$w(S,\bar{S}) = \sum_{i \in S, j \in \bar{S}} w(j,i)$$

A *min-cut* is a cut of minimum weight over all possible cuts; the *min-cut* problem is that of computing the value of the min-cut. Usually we also mean that we want to output some cut of this value. This doesn't make the problem much harder: if you know the min-cut value is c, try removing some edge e, of weight say w. If the remaining graph has a cut of weight $c - w$, then you can safely put this edge in the cut, and if the remaining graph has only cuts of weight $> c - w$, then you can safely exclude the edge from the cut. Either way, you can just check the next edge, in the first case with the graph $G - e$, in the second case with the graph G. (This is known as a "self-reducibility" argument. We saw this idea last term when we were studying algorithms for perfect matching.)

Today: Randomized poly-time algorithm for min-cut. (Deterministic poly-time is known but is much more complicated.) Then we'll combine these ideas (slightly extended) with the #DNF approximation algorithm (also slightly extended), to give a FPRAS (will be defined below) for the network reliability problem.

Contrast: The max-cut problem is NP-complete.

Definition 31. *Let $\{i,j\}$ be an edge of G. In the* contraction *of G by $\{i,j\}$, $G/\{i,j\}$, the vertices i and j are replaced by a single new vertex (i,j), and for each $v \notin \{i,j\}$ any edges $\{i,v\}$ or $\{j,v\}$ are replaced by the edge $\{(i,j),v\}$, with the sum of the constituent weights; the edge $\{i,j\}$ is removed; the rest of the graph remains unchanged.*



With each contraction, the number of vertices of G decreases by one. There is a $1-1$ correspondence between cuts of G that don't separate i and j, and cuts of $G/(i,j)$. In particular, every cut in the graph $G/\{i,j\}$ is a cut in G. So min-cut$(G/\{i,j\}) \geq$ min-cut(G).

Let c be the value of a min-cut of G. In particular, the edges incident on any vertex of G sum to at least c. This remains true of every vertex of H_t (because the min-cut is nondecreasing, as just noted), so

$$w(H_t) \geq \frac{(n-t)c}{2} \tag{3.22}$$

(the factor of two for counting weights from both ends).

Theorem 32. *Let $(S,\bar{S})$ be a min-cut. The probability that Karger's algorithm outputs a refinement of $(S,\bar{S})$ is at least $\frac{n'(n'-1)}{n(n-1)}$.*

Algorithm 4: Karger(G, n'): Karger's Min Cut Algorithm [25]

Input: Undirected simple graph G with edge-weights w; desired size $n' \geq 2$.
Output: An n'-way cut
$H_0 := G$
$t := 0$
while $t < n - n'$ **do**
  Pick $(i, j) \in E(H_t)$ with probability $w(i, j)/w(H_t)$
  $H_{t+1} := H_t / \{i, j\}$
  $t \leftarrow t + 1$
Return the cut of G corresponding to the $n' = n - t$ vertices in H_t.

Exercise: For $n' = 2$ this is tight.

Proof. As noted, $w(S, \bar{S}) = c$. $(S, \bar{S})$ is output by the algorithm if and only if none of the edges crossing this cut is contracted by the algorithm in its $n - 2$ iterations. Suppose that none of the edges in $(S, \bar{S})$ was contracted in H_{t-1}. Then

$$\Pr(\text{an edge of } (S, \bar{S}) \text{ is contracted in } H_t) = \frac{c}{w(H_{t-1})} \leq \frac{2}{n - t + 1}$$

where in the inequality we have applied (3.22). Therefore, for the output of the algorithm,

$$\begin{aligned}
\Pr[\text{min-cut } (S, \bar{S}) \text{ coarsens the output } H_{n-n'}] &= (1 - \frac{c}{w(H_0)})(1 - \frac{c}{w(H_1)}) \cdots (1 - \frac{c}{w(H_{n-(n'+1)})}) \\
&\geq (1 - \frac{2}{n})(1 - \frac{2}{n-1}) \cdots (1 - \frac{2}{n'+1}) \\
&= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \cdots \cdot \frac{n'}{n'+2} \cdot \frac{n'-1}{n'+1} \\
&= \frac{n'(n'-1)}{n(n-1)}. \qquad (3.23)
\end{aligned}$$

□

If this is run to completion ($n' = 2$), the bound is $\frac{1}{\binom{n}{2}}$. This is therefore a lower bound on the probability of success, even if there is only one min-cut.

Corollary 33. *Repeating Karger's algorithm $O(n^2)$ times gives a probability bounded away from 0 of correctly outputting the min-cut value, and repeating the algorithm $O(n^2 \log n)$ times gives a probability bounded away from 0 that we observe* all *min cuts. (Which is even more than we required for this problem – but we'll want the stronger property a little later.)*

The second part of the corollary comes from the well-known:

Coupon collector's problem: sample with repetition from k kinds of coupons. How many trials until all kinds have been seen? (Was assigned as an exercise)

1	2	3	....	k

If all probabilities are $\frac{1}{k}$, then the expected number of trials is $\Theta(k \log k)$.

If all probabilities are at least p, then the expected number of trials is $O(\frac{1}{p} \log \frac{1}{p})$.

It will be a homework problem that each contraction step can be implemented in time $O(n)$, and therefore that one trial of the algorithm runs in time $O(n^2)$. Consequently the time to success with constant probability is $O(n^4)$. Now let's see a faster method.