1.5 Lecture 5 (12/Oct): More on tail events: Kolmogorov 0-1, random walk

A beautiful fact about tail events is Kolmogorov's famous 0-1 law.

Theorem 10 (Kolmogorov) *If B_i is a sequence of independent events and C is a tail event of the sequence, then* $\Pr(C) \in \{0, 1\}$.

We won't be using this theorem, and its usual proof requires some measure theory, so I'll merely offer a few examples of its application.

Bond percolation

Fix a parameter $0 \leq p \leq 1$. Start with a fixed infinite, connected, locally finite graph H, for instance the grid graph $\mathbb{Z}^2$ (nodes (i, j) and (i', j') are connected if $|i - i'| + |j - j'| = 1$) and form the graph G by including each edge of the grid in G independently with probability p. "Locally finite" means the degree of every vertex is finite. The graph is said to "percolate" if there is an infinite connected component.

Percolation is a tail event (with respect to the events indicating whether each edge is present): consider the effect of adding or removing just one edge. Now induct on the number of edges added or removed.

It is easy to see by a coupling argument that Pr(percolation) is monotone nondecreasing in p, as follows: Instead of choosing just a single bit at each edge e, choose a real number $X_e \in [0, 1]$ uniformly. Include the edge if $X_e < p$. Now, if $p < p'$, we can define two random graphs $G_p, G_{p'}$, each is a percolation process from the respective parameter value, and $G_p \subseteq G_{p'}$.

Due to the 0-1 law, there exists a "critical" p_H such that $\Pr(\text{percolation}) = 0$ for $p < p_H$ and $\Pr(\text{percolation}) = 1$ for $p > p_H$. (See Fig. 1.5.) A lot of work in probability theory has gone into determining values of p_H for various graphs, and also into figuring out whether Pr(percolation) is 0 or 1 at p_H.

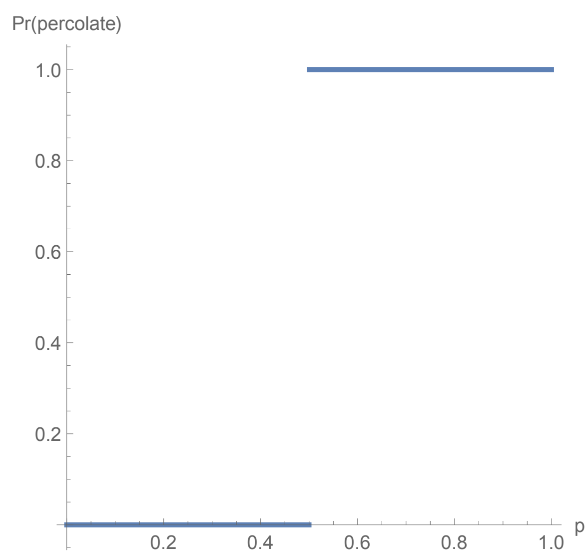

Figure 1.2: Bond percolation in the 2D square grid

Another example of a tail event for bond percolation, this one not monotone, is the event that there are infinitely many infinite components.

No matter what the underlying graph is, the probability of this event is 0 at $p \in \{0, 1\}$. However there are graphs where this probability is 1 at some values in between.

Site percolation

A closely related process is that starting from a fixed infinite, connected, locally finite graph H, we retain *vertices* independently with probability p. (And of course we retain an edge if both its vertices are retained.)

Let N_0 be the random variable representing the *number* of infinite components in the graph that is our random variable. Here "number" can be any nonnegative integer or ∞.

It is known under fairly general conditions (particularly, H should be vertex-transitive), that for any p, exactly one of the following three events has probability 1: $N_0 = 0$; $N_1 = 1$; $N_\infty = 1$. See [57] for the beginning of this story, and [8] for a survey.

Random Walk on $\mathbb{Z}$

Here is another example of a tail event, but this one we can work out without relying on the 0-1 law, and also see which of $0, 1$ is the value:

Consider rw on $\mathbb{Z}$ that starts at 0 and in every step with probability p goes left, and with probability $1 - p$ goes right. Let $L =$ the event that the walk visits every $x \leq 0$. Let $R =$ the event that the walk visits every $x \geq 0$. Each of L and R is a tail event. So by Theorem 10, for any p, $\Pr(L)$ and $\Pr(R)$ lie in $\{0, 1\}$. In fact, we will show—without relying on Theorem 10, but relying on Lemma 8 (Borel-Cantelli I)—that:

Theorem 11

- *For* $p < 1/2$, $\Pr(L) = 0$ *and* $\Pr(R) = 1$.
- *For* $p > 1/2$, $\Pr(L) = 1$ *and* $\Pr(R) = 0$. *(Obviously this is symmetric to the preceding.)*
- *For* $p = 1/2$, $\Pr(L) = \Pr(R) = 1$.

  (Note that if $L \cap R$ occurs, then the walk must actually visit every point infinitely often. (Suppose not, and let t be the last time that some site y was visited. Then on one side of y, the point $t + 1$ steps away cannot have been visited yet, and will never be visited.) Thus in this case of the theorem, since $\Pr(L \cap R) = 1$ *by union bound,* $\Pr($*every point in* $\mathbb{Z}$ *is visited infinitely often*$) = 1$. *The term for this is that unbiased rw on the line is recurrent.)*

Proof: First, no matter what p is, let q_y be the probability that the walk ever visits the point y.

Let's start with the cases $p \neq 1/2$. The first step of the argument doesn't depend on the sign of $p - 1/2$: Consider any y and let $B_{y,t} =$ the event that the walk is at y at time t. The following calculation shows that for any y, $\sum_t \Pr(B_{y,t}) < \infty$: For t s.t. $t = y \bmod 2$, we have

$$\begin{aligned}
\Pr(B_{y,t}) &= \binom{t}{\frac{t-|y|}{2}} p^{\frac{t-y}{2}} (1-p)^{\frac{t+y}{2}} \\
&= \binom{t}{\frac{t-|y|}{2}} \left(\frac{1-p}{p}\right)^{y/2} (p(1-p))^{t/2} \\
&\leq 2^t \left(\frac{1-p}{p}\right)^{y/2} (p(1-p))^{t/2} \\
&= \left(\frac{1-p}{p}\right)^{y/2} (4p(1-p))^{t/2}
\end{aligned}$$

Therefore

$$\sum_t \Pr(B_{y,t}) \leq \left(\frac{1-p}{p}\right)^{y/2} \frac{1}{1-\sqrt{4p(1-p)}}$$

which is $< \infty$ for $p \neq 1/2$. So by Borel-Cantelli-I (Lemma 8), with probability 1, y is visited only finitely many times. Then by the union bound, with probability 1 *every* y is visited only finitely many times.

Now let's suppose further that $p > 1/2$ (i.e., the walk drifts left). Then for any $x \in \mathbb{Z}$,

$$\sum_{y \geq x} \sum_t \Pr(B_{y,t}) \leq \left(\frac{1-p}{p}\right)^{x/2} \cdot \frac{1}{\left(1-\sqrt{(1-p)/p}\right)} \cdot \frac{1}{\left(1-\sqrt{4p(1-p)}\right)} < \infty$$

So we get the even stronger conclusion, again by BC-I, that with probability 1 the walk spends only finite time in the interval $[x, \infty]$. Since this holds for all x, we get $\Pr(L) = 1$. Plugging in $x = 0$ gives $\Pr(R) = 0$.

Applying symmetry, we've covered the first two cases of the theorem.

For $p = 1/2$, the claims $\Pr(L) = 1$ and $\Pr(R) = 1$ are equivalent so let's focus on the first.

The claim $\Pr(L) = 1$ is equivalent to saying that for any $x \geq 0$, with probability 1 the walk reaches the point $-x$. This is the same as saying that in the gambler's ruin problem, no matter what the initial stake x of the gambler, he will with probability 1 go broke.

For $x \geq 0$ let's write $q_x =$ the probability the gambler goes broke from initial stake x. We claim that q_x is *harmonic* on the nonnegative axis with boundary condition $q_0 = 1$. The harmonic condition means that on all *interior points* of the nonnegative axis, which means all $x > 0$, the function value is the average of its neighbors:

$$q_x = (q_{x-1} + q_{x+1})/2$$

That this is so is obvious from the description of the gambler's ruin process. But this equation indicates that q_x is affine linear on $x \geq 0$, because for $x \geq 1$, taking "discrete first derivative", we have:

$$(q_{x+1} - q_x) - (q_x - q_{x-1}) = q_{x+1} - 2q_x + q_{x-1} = 0$$

However, the function q_x is also bounded in $[0, 1]$. So it can only be a constant function, agreeing with its boundary value $q_0 = 1$. □