5.2 Lecture 27 (5/Dec): Applications and further versions of the local lemma

5.2.1 Graph Ramsey lower bound

Ramsey’s theorem (the upper bound on the Ramsey number) runs in the opposite direction to Property B because it establishes the existence of something monochromatic. Not surprisingly, then, our use of the local lemma will be to provide a lower bound on Ramsey numbers. We already saw such a lower bound: a simple union bound argument gave $R(k, k) \geq (1 - o(1)) \frac{k}{e} \sqrt{2} \frac{2^{k/2}}{k}$. Now we will see how to improve this.

**Theorem 93** $R(k, k) \geq \max \{n : e\binom{k}{2}\binom{n}{k-2} \leq 2^{\binom{k}{2} - 1}\}.$ Thus $R(k, k) \geq (1 - o(1)) \frac{k}{e} \sqrt{2} 2^{k/2}$.

To see that the condition on $n$ implies the conclusion, raise each side to the power $\frac{1}{k-2}$. The $e\binom{k}{2}$ and $−1$ are inconsequential; we find that if $n$ satisfies the following, then $R(k, k) \geq n$:

$$(1 + o(1))ne^{-\frac{(k-2)\log(k-2)+k-2}{k-2}} \leq 2^{\frac{2^{k-1}}{k-2}}.$$  

$$ne/k \leq (1 + o(1))2^{\frac{k+1}{k}}.$$

**Proof:** As before, sample a graph from $G(n, 1/2)$. For each set of $k$ vertices the “bad event” of a clique or independent set occurs with probability $2^{\binom{k}{2}}$. For the dependency graph, connect two subsets if they intersect in at least two vertices. The degree of this dependency graph is strictly less than $e\binom{k}{2}\binom{n}{k-2}$ (relying on $k \geq 3$, since the theorem is easy for $k = 2$) because this counts neighbors with the extra information of a distinguished edge in the intersection, so $\Delta + 1 \leq \binom{k}{2}\binom{n}{k-2}$. □

This bound, due to Spencer [96], improves on the union bound by a factor of only 2. While the improvement factor is very small, qualitatively it is meaningful. It shows that a certain negative correlation among edges is possible: you have a graph which is big enough to have on average many copies of each graph of size $k$. (The union bound was tailored so that the expected number of copies of a $k$-clique was just below $1/2$, and the same for a $k$-indep-set. Now we have twice as many places to put each of the $k$ vertices, so we expect to see about $2^{k-1}$ of each of these subgraphs.) Yet as you look across different subgraphs of this graph, there is a kind of negative correlation which prevents the occurrence of these extreme graphs (the $k$-clique and the $k$-indep-set). The FKG inequality helps illustrate that the Lovász local lemma did something truly non-local in the probability space. Any independent sampling method would result in a monotone event such as a specific $k$-clique, being at least as likely as the product of its constituent events (the indicators of each edge in the clique).

It is a major open problem to improve on either $\liminf \frac{1}{k} \log R(k, k) \geq \sqrt{2}$ or $\limsup \frac{1}{k} \log R(k, k) \leq 4$. Actually this gap is small by the standards of Ramsey theory. For more on the general topic see [23].

5.2.2 van der Waerden lower bound

Here is another “eventual inevitability” theorem; as before, the local lemma will provide a counterpoint.

**Theorem 94 (van der Waerden [103])** For every integer $k \geq 1$ there is a finite $W(k)$ such that every two-coloring of $\{1, \ldots, W(k)\}$ contains a monochromatic arithmetic sequence of $k$ terms.
The best upper bound on $W(k)$, due to Gowers [47], is $\leq 2^{2^{2^{2k+9}}}$.

The gap in our knowledge for this problem is even worse than for the graph Ramsey numbers: the current lower bound, which we’ll see below, is $W(k) \geq \frac{2^{k-1}}{(k+2)e^2}$. (A better bound is known for prime $k$.) First we show an elementary lower bound:

**Theorem 95** $W(k) > 2^{\frac{k-1}{k+1}}\sqrt{k - 1}$.

**Proof:** Color uniformly iid. The probability of any particular sequence being monochromatic is $2^{1-k}$. The union bound shows that all these events can be avoided provided

$$\frac{n(n-1)}{k-1}2^{1-k} < 1 \quad (5.3)$$

(count $n$ places the sequence can start, while the step size is bounded by $\frac{n-1}{k-1}$). The bound $n \leq 2^{\frac{k-1}{k+1}}\sqrt{k - 1}$ implies $W(k) > 2^{\frac{k-1}{k+1}}\sqrt{k - 1}$. \hfill $\Box$

Now for the improved bound through the local lemma:

**Theorem 96 (Lovász [33])** $W(k) \geq \frac{2^{k-1}}{(k+2)e^2}$.

**Proof:** Again color uniformly iid. For a dependency graph, connect any two intersecting sequences. The degree of this graph is bounded by

$$\frac{(n-1)k^2}{k-1}$$

($k^2$ choices for which elements they have in common, $\frac{n-1}{k-1}$ possible step sizes). Thus all the bad events can be avoided if

$$2^{1-k} \leq \frac{1}{e(1 + \frac{k^2(n-1)}{k-1})},$$

which in turn is implied by the bound in the statement of the lemma.

The improvement here came because a union bound over approximately $n^2/k$ terms was replaced by a smaller factor of about $nk$. \hfill $\Box$

### 5.2.3 Heterogeneous events and infinite dependency graphs

There are two generalized forms of the local lemma that come fairly easily.

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2The original bound of van der Waerden is of Ackermann type growth [2]. The first primitive recursive bound, due to Shelah [33], is this. For any function $f : \mathbb{N} \to \mathbb{N}$ let $\hat{f} : \mathbb{N} \to \mathbb{N}$ be $\hat{f}(1) = f(1), \hat{f}(k) = f(\hat{f}(k-1)) (k > 1)$. So, letting $\exp_2(k) := 2^k$, the tower function is $T = \exp_2$. Shelah’s bound is of the form $\hat{T}$ or in other words $\exp_2$. 

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Heterogeneous events

It is not necessary that we use the same upper bound on $\Pr(B_j)$ for all $j$. Instead, we can allow events of various probabilities. Those which are more likely to occur, must have in-edges from events of smaller total probability. On the other hand less likely events can tolerate more in-edges (as measured by total probability). This is formulated, in a slightly circuitous way, in the following version of the lemma.

**Lemma 97** Let events $B_j$ and dependency edges $E$ be as before. If there are $x_j < 1$ such that for all $j$,

$$\Pr(B_j) \leq x_j \prod_{(k,j) \in E} (1 - x_k)$$

Then

$$\Pr(\bigcap B_j^c) \geq \prod_j (1 - x_j).$$

The proof method is the same. Show inductively on $m$ that (for any subcollection of $m$ events and any ordering on them), $\Pr \left( B_m \mid \bigcap_{j \leq m-1} B_j^c \right) \leq x_m$.

Infinite dependency graphs

Typically, the restriction that $S$ is finite can be dropped due to compactness of the probability space. Specifically, suppose that—as in most applications—there is an underlying space of independent rvs $X_k$, $k$ ranging over some index set $U$, and each $X_k$ ranging in some compact topological space $R_k$. Moreover suppose that every one of the bad events $B_j$ is a function of only finitely many of the $X_k$’s, say of $k \in U_j \subseteq U$, $U_j$ finite. Suppose moreover that each $B_j$ is an open set in $\prod_{k \in U_j} R_k$. Then each $B_j^c$ is a closed set in the product topology on $\prod_{k \in U} R_k$. Since the product topology is itself compact by Tychonoff’s theorem, it satisfies the Finite Intersection Property: a collection of closed sets of which any finite subcollection has nonempty intersection, has nonempty intersection. Consequently, under the additional topological assumptions made here—which are trivially satisfied if each $X_k$ takes on only finitely many values—the supposition in the local lemma (in either formulation 90 or 97) that $S$ is finite, may be dropped.