5.2 Lecture 27 (5/Dec): Applications and further versions of the local lemma

5.2.1 Graph Ramsey lower bound

Ramsey's theorem (the upper bound on the Ramsey number) runs in the opposite direction to Property B because it establishes the existence of something monochromatic. Not surprisingly, then, our use of the local lemma will be to provide a *lower bound* on Ramsey numbers. We already saw such a lower bound: a simple union bound argument gave $R(k,k) \geq (1-o(1))\frac{k}{e\sqrt{2}}2^{k/2}$. Now we will see how to improve this.

Theorem 93 $R(k,k) \geq \max\{n : e\binom{k}{2}\binom{n}{k-2} \leq 2^{\binom{k}{2}-1}\}$. *Thus* $R(k,k) \geq (1-o(1))\frac{k\sqrt{2}}{e}2^{k/2}$.

To see that the condition on n implies the conclusion, raise each side to the power $\frac{1}{k-2}$. The e, $\binom{k}{2}$ and -1 are inconsequential; we find that if n satisfies the following, then $R(k,k) \geq n$:

$$(1+o(1))ne^{\frac{-(k-2)\log(k-2)+k-2}{k-2}} \leq 2^{\frac{k^2-k}{2(k-2)}}$$

$$ne/k \leq (1+o(1))2^{\frac{k+1}{2}}$$

Proof: As before, sample a graph from $G(n,1/2)$. For each set of k vertices the "bad event" of a clique or independent set occurs with probability $2^{1-\binom{k}{2}}$. For the dependency graph, connect two subsets if they intersect in at least two vertices. The degree of this dependency graph is strictly less than $\binom{k}{2}\binom{n-2}{k-2}$ (relying on $k \geq 3$, since the theorem is easy for $k=2$) because this counts neighbors with the extra information of a distinguished edge in the intersection, so $\Delta + 1 \leq \binom{k}{2}\binom{n}{k-2}$. □

This bound, due to Spencer [96], improves on the union bound by a factor of only 2. While the improvement factor is very small, qualitatively it is meaningful. It shows that a certain negative correlation among edges is possible: you have a graph which is big enough to have on average *many* copies of each graph of size k. (The union bound was tailored so that the expected number of copies of a k-clique was just below $1/2$, and the same for a k-indep-set. Now we have twice as many places to put each of the k vertices, so we expect to see about 2^{k-1} of each of these subgraphs.) Yet as you look across different subgraphs of this graph, there is a kind of negative correlation which prevents the occurrence of these extreme graphs (the k-clique and the k-indep-set).

The FKG inequality helps illustrate that the Lovász local lemma did something truly non-local in the probability space. Any independent sampling method would result in a monotone event such as a specific k-clique, being at least as likely as the product of its constituent events (the indicators of each edge in the clique).

It is a major open problem to improve on either $\liminf \frac{1}{k}\log R(k,k) \geq \sqrt{2}$ or $\limsup \frac{1}{k}\log R(k,k) \leq 4$. Actually this gap is small by the standards of Ramsey theory. For more on the general topic see [23].

5.2.2 van der Waerden lower bound

Here is another "eventual inevitability" theorem; as before, the local lemma will provide a counterpoint.

Theorem 94 (van der Waerden [103]) *For every integer $k \geq 1$ there is a finite $W(k)$ such that every two-coloring of $\{1,\ldots,W(k)\}$ contains a monochromatic arithmetic sequence of k terms.*

The best upper bound on $W(k)$, due to Gowers [47], is[2]

$$W(k) \leq \underbrace{2^{2^{2^{2^{2^{k+9}}}}}}_{\text{five two's}}.$$

The gap in our knowledge for this problem is even worse than for the graph Ramsey numbers: the current lower bound, which we'll see below, is $W(k) \geq \frac{2^{k-1}}{(k+2)e}$. (A better bound is known for prime k.) First we show an elementary lower bound:

Theorem 95 $W(k) > 2^{\frac{k-1}{2}}\sqrt{k-1}$.

Proof: Color uniformly iid. The probability of any particular sequence being monochromatic is 2^{1-k}. The union bound shows that all these events can be avoided provided

$$\frac{n(n-1)}{k-1}2^{1-k} < 1 \tag{5.3}$$

(count n places the sequence can start, while the step size is bounded by $\frac{n-1}{k-1}$). The bound $n \leq 2^{\frac{k-1}{2}}\sqrt{k-1}$ implies 5.3. □

Now for the improved bound through the local lemma:

Theorem 96 (Lovász [33]) $W(k) \geq \frac{2^{k-1}}{(k+2)e}$.

Proof: Again color uniformly iid. For a dependency graph, connect any two intersecting sequences. The degree of this graph is bounded by

$$\frac{(n-1)k^2}{k-1}$$

(k^2 choices for which elements they have in common, $\frac{n-1}{k-1}$ possible step sizes). Thus all the bad events can be avoided if

$$2^{1-k} < \frac{1}{e(1+\frac{k^2(n-1)}{k-1})},$$

which in turn is implied by the bound in the statement of the lemma.

The improvement here came because a union bound over approximately n^2/k terms was replaced by a smaller factor of about nk. □

5.2.3 Heterogeneous events and infinite dependency graphs

There are two generalized forms of the local lemma that come fairly easily.

[2]The original bound of van der Waerden is of Ackermann type growth [2]. The first primitive recursive bound, due to Shelah [93], is this. For any function $f : \mathbb{N} \to \mathbb{N}$ let $\widehat{f} : \mathbb{N} \to \mathbb{N}$ be $\widehat{f}(1) = f(1)$, $\widehat{f}(k) = f(\widehat{f}(k-1))$ $(k > 1)$. So, letting $\exp_2(k) := 2^k$, the tower function is $T = \widehat{\exp_2}$. Shelah's bound is of the form $\widehat{T}$ or in other words $\widehat{\widehat{\exp_2}}$.

Heterogeneous events

It is not necessary that we use the same upper bound on $\Pr(B_j)$ for all j. Instead, we can allow events of various probabilities. Those which are more likely to occur, must have in-edges from events of smaller total probability. On the other hand less likely events can tolerate more in-edges (as measured by total probability). This is formulated, in a slightly circuitous way, in the following version of the lemma.

Lemma 97 *Let events B_j and dependency edges E be as before. If there are $x_j < 1$ such that for all j,*

$$\Pr(B_j) \leq x_j \prod_{(k,j)\in E} (1 - x_k)$$

Then

$$\Pr(\bigcap B_j^c) \geq \prod_j (1 - x_j).$$

The proof method is the same. Show inductively on m that (for any subcollection of m events and any ordering on them), $\Pr\left(B_m \mid \bigcap_{j \leq m-1} B_j^c\right) \leq x_m$.

Infinite dependency graphs

Typically, the restriction that $\mathcal{S}$ is finite can be dropped due to compactness of the probability space. Specifically, suppose that—as in most applications—there is an underlying space of independent rvs X_k, k ranging over some index set U, and each X_k ranging in some compact topological space R_k. Moreover suppose that every one of the bad events B_j is a function of only finitely many of the X_k's, say of $k \in U_j \subset U$, U_j finite. Suppose moreover that each B_j is an *open set* in $\prod_{k\in U_j} R_k$. Then each B_j^c is a closed set in the product topology on $\prod_{k\in U} R_k$. Since the product topology is itself compact by Tychonoff's theorem, it satisfies the Finite Intersection Property: a collection of closed sets of which any finite subcollection has nonempty intersection, has nonempty intersection. Consequently, under the additional topological assumptions made here—which are trivially satisfied if each X_k takes on only finitely many values—the supposition in the local lemma (in either formulation 90 or 97) that S is finite, may be dropped.