2.5 Lecture 12 (29/Oct.): Isolating lemma, finding a perfect matching in parallel

2.5.1 Proof of the isolating lemma

Proof: Write $R_i = \{u_i(1), \ldots, u_i(|R_i|)\}$ with $u_i(j) < u_i(j+1)$ for all $1 \leq j \leq |R_i| - 1$.

Think of u as the mapping $\prod u_i$ where $u_i(j)$ is the evaluation of the function u_i at $1 \leq j \leq |R_i|$.

Let $V = \prod_1^m \{1, \ldots, |R_i|\}$ and $V' = \prod_1^m \{2, \ldots, |R_i|\}$. Any composition $u \circ v$ is a weight function on A, and if $v \in V'$ then this weight function avoids using the weights $u_i(1)$.

Note $|V'|/|V| = \prod_1^m (1 - 1/|R_i|) \geq (1 - 1/r)^m$.

Given $v \in V'$, fix a set $T \in \min(\mathcal{S} : u \circ v)$ of largest cardinality. Define $\phi : V' \to V$ by

$$\phi_v(i) = \begin{cases} v(i) - 1 & : i \in T \\ v(i) & : \text{otherwise} \end{cases}$$

We claim that $\min(\mathcal{S} : u \circ \phi_v) = \{T\}$ and that ϕ is a bijection. Observe that for any $S \in \mathcal{S}$,

$$(u \circ v)(S) - (u \circ \phi_v)(S) = \sum_{i \in S \cap T} (u_i(v(i)) - u_i(v(i) - 1)).$$

with every summand on the RHS being positive. In particular $(u \circ v)(T) - (u \circ \phi_v)(T)$ is the largest change in weight possible for any S, and is achieved by S only if $T \subseteq S$.

Because T has largest cardinality among sets in $\min(\mathcal{S} : u \circ v)$, no other set of $\min(\mathcal{S} : u \circ v)$ can contain T, and therefore T decreases its weight by strictly more than any other set of $\min(\mathcal{S} : u \circ v)$. Other sets of $\mathcal{S}$ might have their weight decrease by the same amount as T, but not more. So, $\min(\mathcal{S} : u \circ \phi_v) = \{T\}$ as desired.

Consequently also T can be identified as the unique min-weight element of $\min(\mathcal{S} : u \circ \phi_v)$. So ϕ can be inverted. (Keep in mind, at different v in the domain in ϕ, different sets T get used, so in order to invert ϕ we need to be able to identify T just from seeing the mapping ϕ_v. (And of course, u, which is fixed.)) Thus $|\phi(V')| = |V'|$. So, with v sampled uniformly,

$$\Pr(|\min(\mathcal{S} : u \circ v)| = 1) \geq \Pr(v \in \phi(V')) = \Pr(v \in V') \geq (1 - 1/r)^m.$$

□

2.5.2 Finding a perfect matching, in RNC

Now we describe the algorithm to find a perfect matching (or report that probably none exists) in a graph $G = (V, E)$ with $n = |V|, m = |E|$.

For every $(i, j) \in E$ pick an integer weight w_{ij} iid uniformly distributed in $\{1, \ldots, 2m\}$. By the isolating lemma, if G has any perfect matchings, then with probability at least $(1 - \frac{1}{2m})^m \geq 1/2$ it obtains a unique minimum weight perfect matching.

Define the matrix T by:

$$T_{ij} = \begin{cases} 0 & \text{if } \{i,j\} \notin E \\ 2^{w_{ij}} & \text{if } \{i,j\} \in E, i < j \\ -2^{w_{ji}} & \text{if } \{i,j\} \in E, i > j \end{cases} \tag{2.6}$$

This is an instantiation of the Tutte matrix, with $x_{ij} = 2^{w_{ij}}$.

Claim 39 *If G has a unique minimum weight perfect matching of G (call it M and its weight $w(M)$) then* $\mathrm{Det}(T) \neq 0$ *and moreover,* $\mathrm{Det}(T) = 2^{2w(M)} \times$ *[an odd number].*

Proof: of Claim: As before we look at the contributions to $\mathrm{Det}(T)$ of all the permutations π that are supported by edges of the graph. The contributions from permutations having odd cycles cancel out—that is just because this is a special case of a Tutte matrix.

It remains to consider permutations π that have only even cycles.

- If π consists of transpositions along the edges of M then it contributes $2^{2w(M)}$.
- If π has only even cycles, but does not correspond to M, then:
  - If π is some other matching M' of weight $w(M') > w(M)$ then it contributes $2^{2w(M')}$.
  - If π has only even cycles and at least one of them is of length ≥ 4, then by separating each cycle into a pair of matchings on the vertices of that cycle, π is decomposed into two matchings $M_1 \neq M_2$ of weights $w(M_1), w(M_2)$, so π contributes $\pm 2^{w(M_1)+w(M_2)}$. Because of the uniqueness of M not both of M_1 and M_2 can achieve weight $w(M)$, so $w(M_1) + w(M_2) > 2w(M)$. □

Now let $\hat{T}_{ij}$ be the (i,j)-deleted minor of T (the matrix obtained by removing the i'th row and j'th column from T), and set

$$\begin{aligned} m_{ij} &= \sum_{\pi:\pi(i)=j} \mathrm{sign}(\pi) \prod_{k=1}^{n} T_{k,\pi(k)} \\ &= \pm 2^{w_{ij}} \,\mathrm{Det}(\hat{T}_{ij}) \end{aligned} \tag{2.7}$$

Claim 40 *For every $\{i,j\} \in E$:*

1. *The total contribution to m_{ij} of permutations π having odd cycles is* 0.
2. *If there is a unique minimum weight perfect matching M, then:*
   (a) *If $\{i,j\} \in M$ then $m_{ij}/2^{2w(M)}$ is odd.*
   (b) *If $\{i,j\} \notin M$ then $m_{ij}/2^{2w(M)}$ is even.*

Proof: of Claim: This is much like our argument for $\mathrm{Det}(T)$ but localized.

1. If π has an odd cycle then it has an even number of odd cycles and hence an odd cycle not containing point i. Pick the "first" odd cycle that does not contain point i and flip it to obtain a permutation π^r. Note that $(\pi^r)^r = \pi$. The contribution of π^r to m_{ij} is the negation of the contribution of π to m_{ij}, because we have replaced an odd number of terms from the Tutte matrix by the same entry with a flipped sign.
2. By the preceding argument, whether or not $\{i,j\} \in M$, we need only consider permutations containing solely even cycles. Just as argued for Claim 39, the contribution of every such permutation π can be written as $2^{w(M_1)+w(M_2)}$, where M_1 and M_2 are two perfect matchings obtained as follows: each transposition (i',j') in π puts the edge $\{i',j'\}$ into both of the matchings; each even cycle of length ≥ 4 can be broken alternatingly into two matchings, one of which (arbitrarily) is put into M_1 and one into M_2.

The only case in which there is a term for which $w(M_1) + w(M_2) = 2w(M)$ is the single case that $\{i, j\} \in M$ and π consists entirely of transpositions along the edges of M. In every other case, at least one of M_1 or M_2 is distinct from M, and therefore $w(M_1) + w(M_2) > 2w(M)$. The claim follows. $\square$

Finally we collect all the elements necessary to describe the algorithm:

1. Generate the weights w_i uniformly in $\{1, \ldots, 2m\}$.
2. Define T as in Eqn (2.6), compute its determinant and if it is nonsingular invert it. (Otherwise, start over.) This determinant computation and the inversion can be done (deterministically) in depth $O(\log^2 n)$ as discussed earlier.
3. Determine $w(M)$ by factoring the greatest power of 2 out of $\mathrm{Det}(T)$.
4. Obtain the values $\pm m_{ij}$ from the equations $m_{ij} = \pm 2^{w_{ij}} \mathrm{Det}(\hat{T}_{ij})$ and

$$\mathrm{Det}(\hat{T}_{ij}) = (-1)^{i+j}(T^{-1})_{ji} \mathrm{Det}(T) \quad \text{(Cramer's rule)}$$

   If $m_{ij}/2^{2w(M)}$ is odd then place $\{i, j\}$ in the matching.
5. Check whether this defines a perfect matching. This is guaranteed if the minimum weight perfect matching is unique. If a perfect matching was not obtained (which will occur for sure if there is no perfect matching, but with probability $\leq 1/2$ if there is one), generate new weights and repeat the process.

Of course, if the graph has a perfect matching, the probability of incurring k repetitions wihout success is bounded by 2^{-k}, and the expected number of repetitions until success is at most 2.

The simultaneous computation of all the m_{ij}'s in step 2 is key to the efficiency of this procedure.

The numbers in the matrix A are integers bounded by $\pm 2^{2m}$. Pan's RNC^2 matrix inversion algorithm will compute A^{-1} using $O(n^{3.5}m)$ processors.

For the maximum matching problem, we use a simple reduction: use weights for each of the non-edges too, but sample those weights uniformly from $2mn + 1, \ldots, 2mn + 2m$ (rather than $1, \ldots, 2m$ like the graph edges). Then no minimum weight perfect matching will use any of the non-edges. The cost of this reduction is that the integers in the matrix now use $O(mn)$ rather than $O(m)$ bits, so the number of processors used by the maximum matching algorithm is $O(n^{4.5}m)$.