1. The Chernoff bounds I showed in class were each derived for particular distributions, so we could write down an mgf explicitly. But sometimes we don’t have that luxury, and need to use a Chernoff bound that relies only on limited information. Here is a case in point.

Let $X_1, \ldots, X_n$ be independent (but not necessarily identically distributed) random variables, each taking values in the interval $[0, 1]$. Let $\overline{X} = \sum X_i/n$ and let $\theta = E(\overline{X})$. Let $\epsilon > 0$.

(a) Show that $\Pr(\overline{X} > (1 + \epsilon)\theta) < \exp(-\theta n((1 + \epsilon) \log(1 + \epsilon) - \epsilon))$.

(b) Show that $\Pr(\overline{X} < (1 - \epsilon)\theta) < \exp(-\theta n\epsilon^2/2)$.

(We used (b) as a lemma in the proof of Bourgain’s $L_1$-embedding theorem. We needed there only the special case that the $X_i$ are Bernoulli rvs.)

Hint: Upper bound the mgf of $X_i$ by that of a Bernoulli distribution.

2. In this problem I’m going to ask you to prove a generalization of the threshold we studied for appearance of $K_4$ in $G(n, p)$. Given a graph $H$, let its “peak density” be

$$\rho(H) = \max_{H' \subseteq H} |E(H')|/|V(H')|$$

where $H' \subseteq H$ denotes that $H'$ is a subgraph of $H$, and where $V$ and $E$ are the vertex and edge sets respectively.

You are to show that $p = n^{-1/\rho}$ is the threshold for appearance of $H$ in $G(n, p)$. To simplify matters, it is enough for this problem if you show that for $\epsilon > 0$, the probability tends to 0 for $p = n^{-1/\rho - \epsilon}$ and to 1 for $p = n^{-1/\rho + \epsilon}$.

Let $h = |V|$ and $e = |E|$. It is certainly possible for $\rho$ to be larger than $e/h$. For example in the “spatula graph” (Fig. 1) with $h = 5, e = 6, p = 5/4$ and the threshold is $n^{-4/5}$ (rather than $n^{-5/6}$ as you might have been tempted to think).

![Figure 1: An abstract spatula](image)

*Hint:* One of the superficial complications in this problem is that the automorphism group of $H$ may be anything from trivial up through all permutations of the vertices (which is what we encountered for $K_4$). For instance the automorphism group of the spatula graph has size 2. This range of possibilities complicates counting how many ways $H$ might turn up in the graph $G$ that we sample from $G(n, p)$. The trick is that, since $H$ is of fixed size, you can afford to count wastefully.

I want you to notice something carefully: there is in this problem a “fake threshold” $n^{-h/e}$. Once $p$ rises above $n^{-h/e}$, the expected number of copies of $X$ in $G$ is $\omega(1)$. Despite this, with probability $1 - o(1)$ there are no copies of $X$ in $G$, until $p$ reaches $n^{-1/\rho}$.
3. Show that the value of the nine-free problem, \( f(n) \) (see previous problem set), is \( O(n^{5/3}) \).
   (In actual fact there is a matching \( \Omega(n^{5/3}) \) bound, but I am not assigning that.)

4. Consider a collection of vectors \( v_i = (x_i, y_i) \) (for \( i = 1, \ldots, n \)), such that for all \( i \), \( x_i \) and \( y_i \) are integer, and \( |x_i|, |y_i| < 2^{n/2}/(c \sqrt{n}) \). For a subset \( I \subseteq [n] \) let \( v_I = \sum_{i \in I} v_i \).
   Show that there is a \( c < \infty \) such that (for any \( n \) and any set of vectors as above) there are disjoint subsets \( I, J \) s.t. \( v_I = v_J \).
   Hint: Sample a uniform random subset \( I \), and (writing \( v_I = (x_I, y_I) \)), consider the variance of \( x_I \).

5. There are two disjoint sets \( A, B \) with \( A \cup B = [n] \) and \( |A| \geq c n \), for some constant \( c > 0 \). Consider sampling a set \( R \subseteq [n] \) as follows (with \( R_i \) denoting the event that \( i \in R \)): the \( R_i \) are pairwise independent, and \( Pr(R_i) = p \) for all \( i \).
   Show that there is a \( d > 0 \) depending only on \( c \) such that for all \( n \) there is a \( p \) such that
   \( Pr((A \cap R \neq \emptyset) \land (B \cap R = \emptyset)) \geq d \).
   How can we use this in regard to Bourgain’s \( L_1 \) embedding?