1. Call a $0 - 1$ matrix “nine-free” if there is no $3 \times 3$ submatrix with all entries one. (Rows and columns need not be consecutive.) Let $f(n)$ denote the maximal number of ones in an $n \times n$ nine-free matrix. Find a lower bound for $f(n)$, i.e., show, for $\alpha$ as large as possible, that there exists a nine-free $n \times n$ matrix $A$ with at least $\alpha$ ones. Hint: Use the deletion method, first letting $P(A_{ij} = 1) = p$ and then changing a one to a zero in every $3 \times 3$ submatrix with all entries one.

2. (a) Let $A$ be a random $n \times n$ matrix with entries chosen independently and uniformly in $\{0,1\}$. Find the expectations of the determinant and the permanent of $A$.

   Note: You are certainly familiar with the first of these concepts, possibly not with the second; they are defined by the formulas

   $$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

   $$\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

   (b) Let $A$ be a random $n \times n$ matrix with entries chosen independently and uniformly in $\{-1,1\}$. Find $E((\det(A))^2)$.

3. A monotone function $f : \{0,1\}^n \to \{0,1\}$ is one with the property that if $f(\ldots, x_{i-1}, 0, x_{i+1}, \ldots) = 1$ then $f(\ldots, x_{i-1}, 1, x_{i+1}, \ldots) = 1$. Here are some examples: the “dictator function” $\text{Dict}_{ij}(x_1, \ldots, x_n) = x_j$; the AND$_n$ function which is 1 only for $x_1 = \ldots = x_n = 1$; the function MAJ$_n$ for odd $n$ which is 1 if more than half of the inputs are 1’s; the function CLIQUE$_{n,k}(x_1,1, x_1,2, \ldots, x_{n-1}, k)$ which is 1 if the graph having an edge for each “1”, contains a clique of size $k$.

When we design boolean circuits for functions, we use a fixed (and constant-size) basis of gates. For instance the basis $\{\text{AND}_2, \text{NOT}\}$, or even just the basis with the single gate $\{\text{NAND}_2\}$. If we are only interested in computing monotone functions, however, then we can consider using a basis consisting only of monotone gates. This is not necessarily the simplest or most efficient way of constructing a circuit. For instance the simplest way to compute MAJ$_n$ is to use a general (non-monotone) basis to perform arithmetic, and add up the input bits and check whether the sum is $> n/2$. In order to understand the power of non-monotonicity, even for computing monotone functions, we need to ask how efficiently we can compute functions like MAJ$_n$ using only a monotone basis. That is what we will do in this exercise.

The basis we consider is simple: it includes only the 3-input gate MAJ$_3$. Your task is to show something not at all obvious: there are log-depth circuits for MAJ$_n$ consisting solely of MAJ$_3$ gates.

Hints:

(a) There exists a circuit of the following simple form: the MAJ$_3$ gates form a complete 3-ary tree from the output gate all the way down to input wires at depth $O(\log n)$. Then each of these wires is randomly, independently, hooked up to one of the $n$ inputs. (Note, each input will be used many times.)

(b) A good approach is to show that for any particular $x = (x_1, \ldots, x_n)$, with very low probability the circuit you constructed at random gives the wrong answer.

(c) For any particular $x$, let $p_t$ be the probability that a wire at level $t$ of the circuit carries a value that disagrees with MAJ$_n(x)$. Show that $p_1 \leq (n-1)/(2n)$ and $p_{t+1} = 3p_t^2 - 2p_t^3$. 
4. The following is an example of a heavy-tailed distribution. $\mu$ is supported on the nonzero integers,

$$\mu(m) = \frac{K}{m^4}$$

for the appropriate normalizing constant $K$ which is $45/\pi^4$.

The first and second moments of $\mu$ are well-defined; if you calculate you’ll see $E(X) = 0$, $\text{Var}(X) = 15/\pi^2$.

The purpose of this exercise is to demonstrate that for a heavy-tailed distribution like this, taking the average of a large number of independent samples does not create a light-tailed distribution. Specifically, take $n$ iid rvs $X_1, \ldots, X_n$ with the distribution $\mu$, and set $\bar{X} = (1/n) \sum X_i$. The second-moment inequality tells us:

$$\Pr(|\bar{X}| \geq \lambda \sqrt{\text{Var}(\bar{X})}) \leq \frac{1}{\lambda^2}$$

(Specifically $\Pr(|\bar{X}| \geq r) \leq \frac{15}{\pi^2 nr^2}$.)

Show that there is a polynomial $p(\lambda, n)$ such that

$$\Pr \left( \bar{X} > \lambda \sqrt{\text{Var}(\bar{X})} \right) \geq 1/p(\lambda, n).$$

What does this tell you about the moment generating function of $\mu$?