Leonard J. Schulman, schulman@caltech.edu **TA: ?, ?@caltech.edu**
Problem set 1 **Out W 8/Oct Due W 15/Oct in class**

1. The first and second Borel-Cantelli lemmas are only weak converses of each other, since the former makes no independence assumption and the latter makes an assumption of total independence of the events. In this problem we shall reduce the gap by showing that pairwise independence—in fact even a bit less than that—is an adequate assumption for the second B-C lemma.

   Pairwise independence of events is like independence except that instead of requiring that for any *finite* collection of events, $\Pr(\cap B_i) = \prod \Pr(B_i)$, we require this only for pairs of events. What we mean by "a bit less than that" is spelled out here:

   Show that if $\sum \Pr(B_i)$ diverges and

   $$\liminf_n \frac{\sum_{j,k \leq n} P(B_j \cap B_k)}{\left(\sum_{j \leq n} \Pr(B_j)\right)^2} \leq 1$$

   then $\Pr(\limsup B) = 1$.

2. The Ramsey Number $R(k, \ell)$ is the least n such that a graph with n vertices must contain either a clique of size k or an independent set of size ℓ. Prove that if there is a real p, $0 \leq p \leq 1$ such that $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1$, then $R(k, \ell) > n$. Using this, show that $R(4, \ell) \in \Omega((\ell / \log \ell)^{3/2})$.

3. Let $[n] = \{1, \ldots, n\}$, and let F be an intersecting family of k-sets of $[n]$. That is, each $A \in F$ is a subset of $[n]$ with $|A| = k$; and for all $A, B \in F$, $A \cap B \neq \emptyset$. In this exercise you are to show that if $n \geq 2k$, $|F| \leq \binom{n-1}{k-1}$.

   Hint. For any permutation σ of U, and any $1 \leq i \leq n$, define the set $A_{\sigma,i} = \{\sigma(i), \ldots, \sigma(i+k-1)\}$ (indices interpreted modulo n). (a) Show that for any fixed σ, at most k of the n sets $A_{\sigma,i}$ can be in F. (b) Select a random $A_{\sigma,i}$ and show that with probability at most k/n it belongs to F; consider what this implies for random k-sets.