Probability and Algorithms **Caltech CS150, Winter 2007**
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Problem set 4 **Out Friday March 2. Due Monday March 12.**

1. **[No collaboration]** Football season is *still* over, and a sports channel is still broadcasting its strangely popular show in which a fair coin is flipped many times and the audience is allowed to place bets on what happens. Recall that for a sequence $w \in \{H,T\}^*$, X_w is the positive integer random variable denoting the index of the first occurrence of w (as a contiguous subsequence) in the sequence of coin tosses.

   In this exercise consider only three-character sequences, namely $w \in \{H,T\}^3$. Show that for every w there exists a w' such that $P(X_{w'} < X_w) > 1/2$.

2. Consider the $(n+1) \times n$ grid graph. (E.g., the 2×1 case is a single edge, the 3×2 case looks like a figure eight in calculator numerals). Think of these graphs as laid out with the short side running horizontally. In the *edge percolation* process with parameter p, each edge is, independently, retained with probability p, or erased with probability $1-p$. Let V be the event that "vertical percolation" occurs, i.e., that any of the vertices in the top row is connected by a path to any of those in the bottom row.

   Show that
   $$P(V > 1/2) \quad \text{if and only if} \quad p > 1/2$$

3. In class I proved that for a self-reducible problem, the following are equivalent:

   (a) $\forall k \; \exists$ a polynomial-time algorithm that samples from a distribution p on the "objects" (the elements of S_P) s.t. if u is the uniform distribution on these objects, then $||p-u||_1 \leqslant n^{-k}$.

   (b) $\forall k \; \exists$ a polynomial-time algorithm that estimates the number of objects (the cardinality of S_P) to within a multiplicative factor $1 \pm n^{-k}$, except for an error event that occurs with probability n^{-k}.

   For the $3b \to 3a$ direction I only sketched the argument; please supply it.

4. **[No collaboration]** Let $1 \leq k \leq n$. Show that in a uniformly random permutation of $\{1,\ldots,n\}$, the expected number of cycles of length k is $1/k$.

5. A set of prisoners numbered 1 through n face a predicament.

   They stand on the bank of a wide river and are being ferried, one by one, to an island in the river. They are all sentenced to life imprisonment there unless *every one of them* succeeds at the following strange task.

   In a cabin on the ferry there are n boxes called $b_1,\ldots,b_n$, containing, in a random permutation, tickets labeled $c_1,\ldots,c_n$. During prisoner i's ferry ride he has time to open $\lceil n/2 \rceil$ of the boxes. His goal is to find ticket c_i; the ferry captain records whether he succeeds. The prisoner cannot remove or mark any of the tickets or boxes; after he is dropped off, the captain shuts all the boxes. No communication between the island and the riverbank is possible.

   Show that the prisoners can achieve a constant probability (independent of n) of gaining their freedom.