Week 6 reading: MU 6.7-6.8, 7.1-7.4, proof of the Perron-Frobenius theorem in notes of Andries Brouwer on the class website.

This assignment is longer than usual so I'm allowing two weeks for it. Please get started early though. I haven't assigned work on the Perron-Frobenius theorem but it is an important theorem with an interesting proof, and well worth learning. (You can also find the proof elsewhere, e.g., some notes online by Mike Boyle.)

Exercises: MU 6.16, 6.18, 7.1, 7.6, 7.8, 7.13, 7.17, 7.19, 7.26.

Also the following two exercises:

(A) Let the *van der Waerden* function $W(k)$, for nonnegative integer k, be the least nonnegative integer n so that if $\{1,\ldots,n\}$ is two-colored, there exists a monochromatic arithmetic progression with k terms. Remarkably, $W(k)$ is finite for all k; the upper bound is immense and the true value is far from known. In this exercise you are asked to show a lower bound on $W(k)$. Try coloring $\{1,\ldots,n\}$ independently, uniformly at random, and consider for each arithmetic progression S the event A_S that S is monochromatic. Use the Lovasz local lemma to show that $W(k) > 2^k/(8k)$.

(B) Consider the infinite graph on the vertex set $\mathbb{Z}^d \times \mathbb{Z}$ (where the last coordinate is interpreted as "time") and with a directed edge from vertex $(x_1,\ldots,x_d,t)$ to vertex $(y_1,\ldots,y_d,t')$ if $t' = t+1$ and $|x_i - y_i| \leq 1$ for all i.

A trajectory γ of length $|\gamma| = t$ and which begins at time t_0 is a mapping from $\{t_0,\ldots,t_0+t\}$ to V for which all $(\gamma(i),\gamma(i+1)) \in E$. If two trajectories γ,γ' are of equal length, start at the same time t_0, and share the same start vertex (i.e., $\gamma(t_0) = \gamma'(t_0)$), we write $\gamma \sim \gamma'$. The graph distance τ between trajectories $\gamma \sim \gamma'$ of length t is $\tau(\gamma,\gamma') = |\{t_0 < i \leq t_0 + t : \gamma(i) \neq \gamma'(i)\}|$.

Let S be a set of labels. A *trajectory code* is a mapping $\mathbb{Z}^d \times \mathbb{Z} \to S$, extended to a mapping from trajectories to S^* by concatenation: $\chi(\gamma) = (\chi(\gamma(t_0+1)),\ldots,\chi(\gamma(t_0+t)))$. Hamming distance between equal-length words in S^* is denoted h. The *relative distance* of the code is defined to be $\delta = \inf_{\gamma\sim\gamma'}\{h(\chi(\gamma),\chi(\gamma'))/\tau(\gamma,\gamma')\}$.

We say that the trajectory code is *asymptotically good* if S is finite and the relative distance δ of the code is positive.

Use the Lovasz local lemma to show that there exists an asymptotically good trajectory code.

(Exercise taken from Ostrovsky, Rabani and Schulman, FOCS'05.)