Probability and Algorithms **Caltech CS150, Winter 2006**
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Problem set 1 **Out Friday January 6. Due Friday January 20.**

Suggested references are listed on the course web page. You're welcome to take advantage of other references. However, realize that some of the problems I'll assign are "standards," whose solutions might be found in textbooks, notebooks of older students, and so forth. Regardless of the source, don't read existing solutions before turning in your work. This includes materials from previous offerings of this course.

Collaboration: Unless a problem is marked "no collaboration," you can work with other students in the class (I encourage it). But you should produce the final write-up for submission alone, after all discussions, and if you worked with someone on a problem, you should list their name by your solution. You can discuss "no collaboration" problems with the TA or with me. I won't usually discuss material over email.

1. **[No collaboration]** Mitzenmacher & Upfal 2.20.
2. M&U 2.26.
3. **[No collaboration]** M&U 3.3.
4. Let $X_1, \ldots, X_n$ be independent rv's with $E(X_i) = 0$ and $|X_i| \leq 1$. Let $\overline{X} = (\sum X_i)/n$ and $\lambda \geq 0$. Show that $\Pr(|\overline{X}| > \lambda/\sqrt{n}) < 2e^{-\lambda^2/2}$.
5. **[No collaboration]** Let $Y_1, \ldots, Y_n$ be independent rv's with $\Pr(Y_i = 1) = p_i, \Pr(Y_i = 0) = 1 - p_i$. Let $Y = \sum Y_i$, $p = (\sum p_i)/n$, and $\beta > 1$. Show that $\Pr(Y > \beta pn) < (e^{\beta-1}\beta^{-\beta})^{pn}$.
6. Show that the d-dimensional Gaussian densities (parameterized by σ), $(\sqrt{2\pi}\sigma)^{-d}e^{-(2/\sigma^2)\sum x_i^2}$, are the only continuously differentiable probability densities in $\mathbb{R}^d$ that are spherically symmetric (invariant under orthogonal transformations) and whose coordinates (the x_i) are statistically independent.
7. **[No collaboration]** Here's a nice example of the *probabilistic method* in combinatorics. There are n^2 light bulbs arranged in a square grid. Initially some are lit and some are not. The initial assignment is specified with variables $\{a_{ij}\}_{i,j\in\{1,\ldots,n\}}$, where $a_{ij} = 1$ if the bulb in row i and column j is on, and $a_{ij} = -1$ if the bulb is off. Now, in each row and each column there is a switch which can simultaneously flip all the bulbs in that row or column. Let x_i denote the switch for row i, where $x_i = 1$ means that the row is left alone, and $x_i = -1$ means that it is flipped. Similarly let $y_j \in \pm 1$ denote the switch for column j.

   We're interested in the following function: $F(n) = \min_{a_{ij}\in\pm 1} \max_{x_i, y_j \in \pm 1} \sum_{i,j} a_{ij} x_i y_j$. In other words, how can the initial assigment be set so as to minimize the maximum number of bulbs that can be lit?

   Show that $F(n) \leq (2 \log^{1/2} 2) n^{3/2}$. (Natural log.)
8. A standard deck of 52 cards has been thoroughly shuffled and placed face down. You turn over cards one at a time until you reach the first ace. What is the probability that the next card is an ace?