Out line – Talk I *

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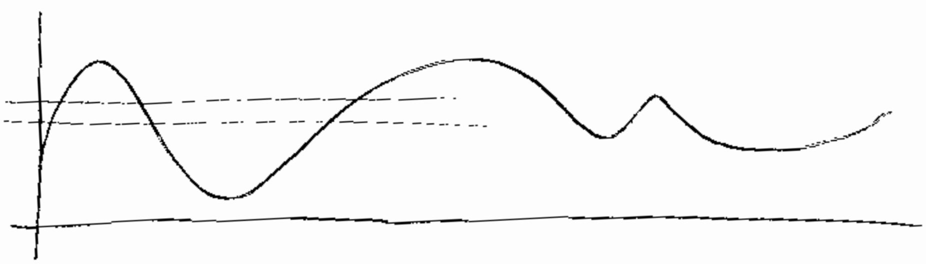
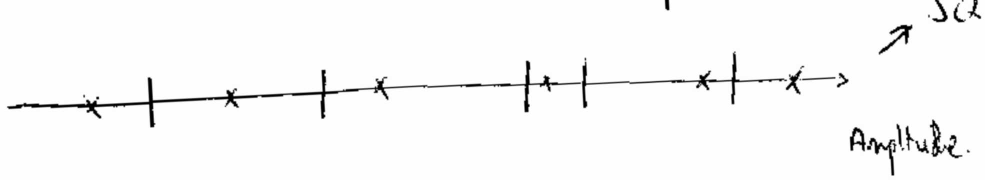
* For more details see Lecture Notes of EECS 651 by David L. Neuhoff at the University of Michigan. http://www.eecs.umich.edu/courses/eecs651/index.htm

CMI Talk I

Introduction: Quantization as lossy data compression



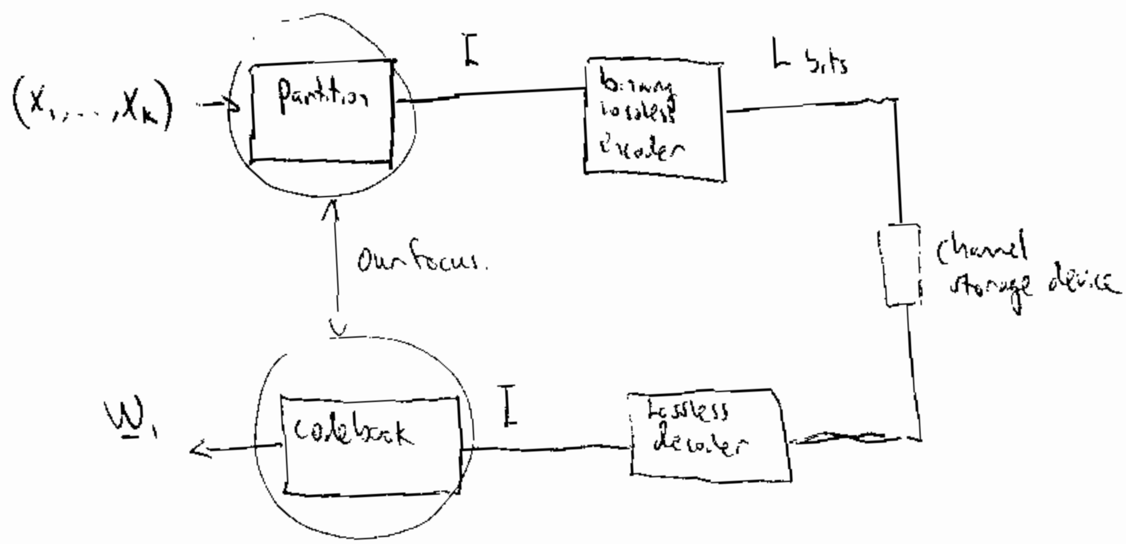
Explain the scalar quantization operation



can take two samples at a time. VQ

Slides: 10-14   2-D q. rule.

General.



quantizer parameters:

K - dimension

M - number of cells

$\mathcal{W} = \{\underline{w}_1, \ldots, \underline{w}_M\}$ - code vectors/points.

$q: \mathbb{R}^k \to \mathcal{W}$ $(X_1, \ldots, X_k) \to Q \to (Y_1, \ldots, Y_k) \in \mathcal{W}$

Performance Measures

Rate: # bits per source input (Rate is determined by lossless encoder)

$$R = \frac{L}{k} \approx \frac{1}{k}\log_2 M$$

Distortion: D = MSE (normalized by dimension).

$$= \frac{1}{k}\sum_{i=1}^{k} E(X_i - Y_i)^2$$

$$= \frac{1}{k} E\,\|\underline{X} - Q(\underline{X})\|^2$$

$$= \frac{1}{k}\int \|\underline{x} - Q(\underline{x})\|^2 f_{\underline{x}}(\underline{x})\,d\underline{x}$$

$$= \frac{1}{k}\sum_{i=1}^{M}\int_{S_i} \|\underline{x} - \underline{w}_i\|^2 f_{\underline{x}}(\underline{x})\,d\underline{x}$$

Distortion determined by partition and codebook

* We consider Fixed-Rate i.e. every quantization cell index is represented by the same # of bits: kR

Goals

1) Find $\delta(k, M)$ OPTA function for arbitrary k and large M.
2) Show that VQ's are better than SQ's (even for IID sources)
3) Understand why
4) Quantify by how much.

High Resolution

* Most cells are small so $f(x) \approx$ constant on each.
* neighboring cells have similar sizes and shapes

We wish to examine how distortion is effected.

- examples of various 2-D quantizers (overhead)

Gross key characteristics of a quantizer:

- dimension k
- number of cells M
- function describing the distribution or density of cells over R^k – point density
- Something to do with the shapes of cells.

Bennetts integral: For high resolution quantizers with congruent cells (same shape, size doesn't matter).

↓ Special case.

Spectrum of quantized Signals.

1948 Bell Sys Tech J.

$$D \cong \frac{1}{M^{2/k}}\, m \int \frac{1}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\,d\underline{x}$$

where

m – a quantity that depends only on the shape of cells.

$\lambda(\underline{x}) \approx$ density of cells approximately near $\underline{x}$.

Gersho 1979
Asymptotically Optimal Block Quantization

point density

Slides. Bennett 9, 10

Properties of $\lambda(\underline{x})$

1) $\int_A \lambda(\underline{x})\,d\underline{x} \approx$ fraction of cells/points in region A.

2) $\lambda(\underline{x}) \cdot |A| \simeq \dfrac{\#\text{ cells in } A}{M}$ when A small but still much larger than cells near $\underline{x}$

3) $\lambda(\underline{x}) \geq 0$, $\int \lambda(\underline{x})\,d\underline{x} = 1$

4) $\lambda(\underline{x}) \cong \dfrac{1}{M|S_i|}$, $\underline{x} \in S_i$

because: $\#\text{ Cells in } A \cong \dfrac{|A|}{\text{cell vol}}$ Assuming most cells have about the same volume in a small region A.

$\lambda(\underline{x}) \cdot |A| = \dfrac{\#\text{ cells in } A}{M} = \dfrac{|A|}{\text{cell volume} \cdot M} \Rightarrow \lambda(\underline{x}) \cong \dfrac{1}{\text{cell volume} \cdot M}$

---

let $\mathcal{M}(S, \underline{y}) = \int_S \|\underline{x} - \underline{y}\|^2 d\underline{x}$ $\rightarrow$ moment of inertia of S about $\underline{y}$
MI

then

$m(S, \underline{y}) = \dfrac{\mathcal{M}(S, \underline{y})}{K \cdot |S|^{\frac{k+2}{k}}}$ = "normalized moment of inertia" of S about $\underline{y}$
NMI

$m(aS, a\underline{y}) = \cdots = m(S, \underline{y}) \Rightarrow$ see back page.

$m(S + \underline{v}, \underline{y} + \underline{v}) = \cdots = m(S, \underline{y})$

$m(S, \underline{y})$ not affected by scaling or translation.
Thus determined by shape not size or position

Slides, Bennett 25,26

$m(S, \underline{y})$ invariant to scaling:

let $a > 0$:

$aS = \{ \underline{z} = a\underline{x} : \underline{x} \in S \}$.

$$m(aS, a\underline{y}) = \frac{\int_{aS} \|\underline{x} - a\underline{y}\|^2 d\underline{x}}{k \cdot |aS|^{\frac{k+2}{k}}} \underset{\substack{a\underline{z} = \\ a^k d\underline{z} = d\underline{x}}}{=} \frac{\int_S \|a\underline{z} - a\underline{y}\|^2}{k \cdot |aS|^{\frac{k+2}{k}}}$$

$$= \frac{\int_S \|\underline{z} - \underline{y}\| \, a^{2+k} d\underline{z}}{k \cdot |S|^{\frac{k+2}{2}} \cdot (a^k)^{\frac{k+2}{k}}} = \frac{\int_S \|\underline{z} - \underline{y}\|^2 d\underline{z}}{k \cdot |S|^{\frac{k+2}{k}}} = m(S, \underline{y})$$

$|aS| = a^k |S|$

example: Scalar quantizer $(k=1)$

Cells are intervals. reproduction points are at cell midpoints (i.e. centers).

$$m(\text{interval}) = \frac{1}{12}.$$

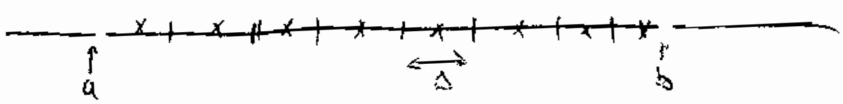
and so,

$$D = \frac{1}{12M^2}\int_{-\infty}^{\infty}\frac{f_X(x)}{\lambda^2(x)}\,dx.$$

Special case: Uniform scalar quantizer

quantizer is uniform over $[a,b]$ i.e.

1) $[a,b]$ divided into M cells of equal size $\Delta = \frac{b-a}{M}$
2) reproduction points at cell centers.



Assume $\Pr(a \le X \le b) \cong 1$.

$$D \cong \frac{1}{12M^2}\int_a^b \frac{f_X(x)}{\frac{1}{(M\Delta)^2}}\,dx = \frac{M^2\Delta^2}{12M^2}\int_a^b f(x)\,dx = \boxed{\frac{\Delta^2}{12}}.$$

where, $\lambda(x) \cong \frac{1}{M\Delta}$

Derivation of Bennett's Integral

$$D = \frac{1}{K} E \| \underline{X} - Q(\underline{X}) \|^2$$

$$= \frac{1}{K} \sum_{i=1}^{M} \int_{S_i} \| \underline{x} - \underline{w}_i \|^2 f_{\underline{X}}(\underline{x}) d\underline{x}$$

$$\cong \frac{1}{K} \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \int_{S_i} \| \underline{x} - \underline{w}_i \|^2 d\underline{x}$$ H.R. assumption Most cells are small.

$$= \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \frac{1}{K} \mathcal{M}(S_i, \underline{w}_i)$$ $\mathcal{M}(S_i, \underline{w}_i)$ is MI of S_i about $\underline{w}_i$

$$= \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\, m(S_i, \underline{w}_i) \cdot |S_i|^{1+2/K}$$ $\mathcal{M}(S_i, \underline{w}_i) = m(S_i, \underline{w}_i) \cdot K\, |S_i|^{1+2/K}$

$$\cong \frac{1}{M^{2/K}} \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \frac{m(\underline{w}_i)}{\lambda^{2/K}(\underline{w}_i)} |S_i|$$ $|S_i| \cong \frac{1}{M \lambda(\underline{w}_i)}$, $m(\underline{w}_i) \cong m(S_i, \underline{w}_i)$

$m(\underline{x})$ = NMI of cell containing $\underline{x}$
"inertial profile."

$$\cong \frac{1}{M^{2/K}} \int \frac{m(\underline{x})}{\lambda^{2/K}(\underline{x})} f_{\underline{X}}(\underline{x}) d\underline{x}$$ by definition of an integral

Note: when cells are very small $D \to 0$. So we actually have

$$\frac{D}{\frac{1}{M^{2/K}} \int \frac{m(\underline{x})}{\lambda^{2/K}(\underline{x})} f_{\underline{X}}(\underline{x}) d\underline{x}} \cong 1$$

Find formula for least distortion of Fixed-Rate VQ, i.e. the OPTA function $\delta(k, M)$

From Bennett's integral we have:

$$D \cong \frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{x}}(\underline{x})\, d\underline{x}$$

Steps:

1) Find best inertial profile
2) Find best point density
3) Substitute into Bennett's integral and obtain OPTA function.

STEP 1:

Claim: $\forall k$ the best inertial profile is a constant

Sketch of proof: By contradiction.

Suppose an optimal Q does not have constant inertial profile. Then $\exists\, \underline{x}, \underline{y}$ s.t. $m(\underline{x}) > m(\underline{y})$. Replace the cells in the vicinity of $\underline{x}$ with the pattern of cells in the vicinity of $\underline{y}$. This reduces distortion, contradicting the optimality of Q. (THIS has never been carefully proven).

Let m^*_k denote the constant.

→ NOT Gersho's Conjecture.

Gersho's Conjecture   1979 Asymptotically Optimal Block Quantization   (6)

* Most cells of an optimal fixed-rate H.R. quantizer are approximately congruent to some basic cell shape (locally tesselations)
  can scale too

* The basic shape is the k-dim tesselating polyhedron with least NMI
  partition of R^k with each cell being a translation/rotation of basic shape. not scaling

m^*_k = least NMI of any tesselating k-dim polyhedron
  - Gersho's constant.

Slide: Zador 34

STEP 2: Best point density

We assume best inertial profile is constant.
We need to find the point density that minimizes the remaining part of Bennett's integral.

$$\int \frac{f_{\underline{X}}(\underline{x})}{\lambda^{2/k}(\underline{x})} \, d\underline{x}$$

It turns out to be: $\lambda_k^*(\underline{x}) = \dfrac{f_{\underline{X}}^{\frac{k}{k+2}}(\underline{x})}{\int f_{\underline{X}}^{\frac{k}{k+2}}(\underline{x}')\, d\underline{x}'}$

and the resulting minimum value is: $\left(\int f_{\underline{X}}^{\frac{k}{k+2}}(\underline{x})\, d\underline{x}\right)^{\frac{k+2}{k}}$

substituting λ_k^*, m_k^* into Bennett's integral

$$\boxed{\delta(k,M) \cong \frac{1}{M^{2/k}}\, m_k^* \left(\int f_{\underline{x}}(\underline{x})^{\frac{k}{k+2}}\, d\underline{x} \right)^{\frac{k+2}{k}}} \longrightarrow \text{Zador's Theorem}$$

1963 (Thesis

$$= \frac{1}{M^{2/k}} \cdot m_k^* \cdot \beta_k \cdot \sigma^2 \quad = \sigma^2 m_k^* \beta_k\, 2^{-2R}$$

where

$$\sigma^2 = \text{source variance} = \frac{1}{k} \sum_{i=1}^{k} \text{Variance}(x_i)$$

$$\beta_k = \frac{1}{\sigma^2} \cdot \left(\int f_{\underline{x}}(\underline{x})^{\frac{k}{k+2}}\, d\underline{x} \right)^{\frac{k+2}{k}}$$

In terms of SNR:

$$\boxed{\begin{aligned} S(k,R) &\cong 10 \log_{10} \frac{\sigma^2}{\delta(k,R)} \\ &= 6.02R - 10\log_{10} m_k^* \beta_k \end{aligned}}$$

SNR increases 6 dB per bit for optimal quantizers

NOTES: 1) Originally Zador had a direct proof that $\forall k$ $\exists$ const α_k s.t. $\forall f_{\underline{x}}(\underline{x})$

$$\lim_{M \to \infty} M^{2/k} \delta(k,M) = \alpha_k \left(\int f_{\underline{x}}(\underline{x})^{\frac{k}{k+2}}\, d\underline{x} \right)^{\frac{k+2}{k}}$$

He didn't equate α_k with m_k^* but gave bounds to it.

2) Accurate for $R \gtrsim 3$ $(M \geq 2^{3k})$

For ... with k

Properties of Optimal Quantizers

- Cell Volume: $|S_{\underline{x}}| \cong \dfrac{1}{M\,\lambda_k^*(\underline{x})} = \dfrac{c}{M\,f_{\underline{X}}(\underline{x})^{\frac{k}{k+2}}}$

  Smaller where f is larger.

- Cell probability: $\Pr(S_{\underline{x}}) \cong f_{\underline{X}}(\underline{x})\,|S_{\underline{x}}| \cong f_{\underline{X}}(\underline{x})\,\dfrac{c}{M}\cdot f_{\underline{X}}(\underline{x})^{-\frac{k}{k+2}} = \dfrac{c}{M}\cdot f_{\underline{X}}(\underline{x})^{\frac{2}{k+2}}$

  Larger where f is larger

- Cell distortion:

Z-45 erroneous slide

$$D_{\underline{x}} = \frac{1}{k}\int_{S_{\underline{x}}} \|\underline{x}'-Q(\underline{x})\|^2 f_{\underline{X}}(\underline{x}')\,d\underline{x}' \cong \frac{1}{k} f_{\underline{X}}(\underline{x}) \int_{S_{\underline{x}}} \|\underline{x}'-Q(\underline{x})\|^2\,d\underline{x}'$$

$$= \frac{1}{k} f_{\underline{X}}(\underline{x})\,k\,m_k^*\,|S_{\underline{x}}|^{\frac{k+2}{k}}$$

$$= f_{\underline{X}}(\underline{x})\,m_k^*\,\frac{c}{M}\,f_{\underline{X}}(\underline{x})^{-\frac{k}{k+2}\cdot\frac{k+2}{k}} = m_k^*\cdot\frac{c}{M}$$

Same for all $\underline{x}$. i.e. all cells contribute the same to distortion

- Conditional cell distortion:

$$\frac{1}{k}\int_{S_{\underline{x}}} \|\underline{x}'-Q(\underline{x})\|^2 f_{\underline{X}}(\underline{x}'\,|\,\underline{X}\in S_{\underline{x}})\,d\underline{x}' = \frac{1}{k}\int_{S_{\underline{x}}} \|\underline{x}'-Q(\underline{x})\|\,\frac{f_{\underline{X}}(\underline{x}')}{\Pr(S_{\underline{x}})}\,d\underline{x}'$$

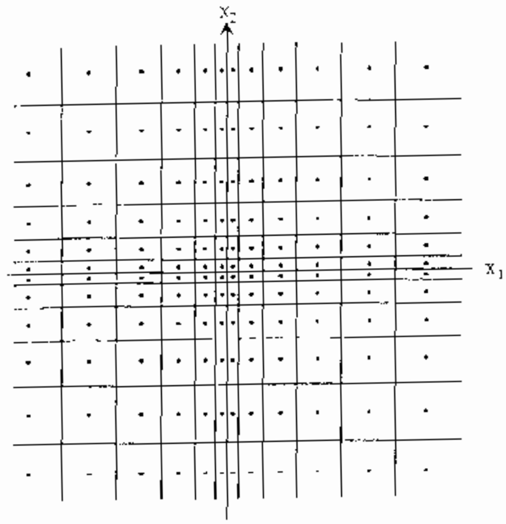
$$\cong \frac{1}{\Pr(S_{\underline{x}})}\cdot m_k^*\cdot\frac{c}{M}$$

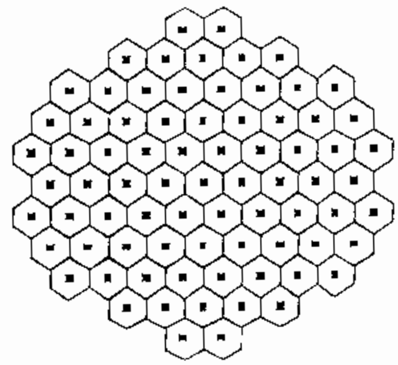
$$= \frac{M}{c}\,f_{\underline{X}}(\underline{x})^{-\frac{2}{k+2}}\cdot m_k^*\,\frac{c}{M}$$

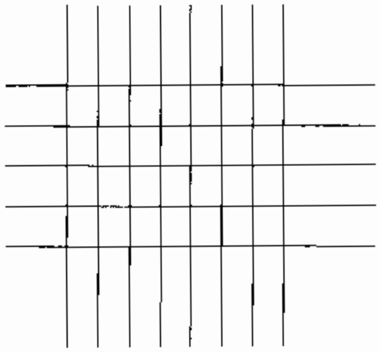
Inversely proportional to cell probability. Smaller where f is larger.

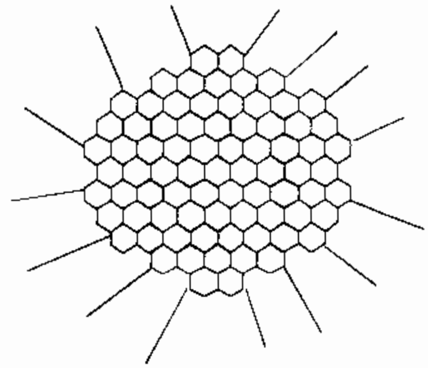
EXAMPLES

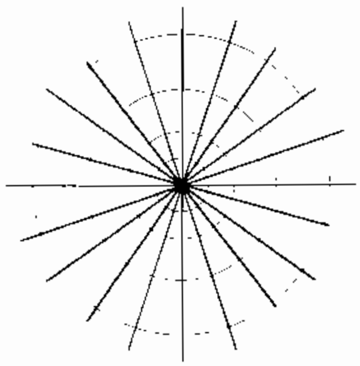
(in k=2 dimensions)

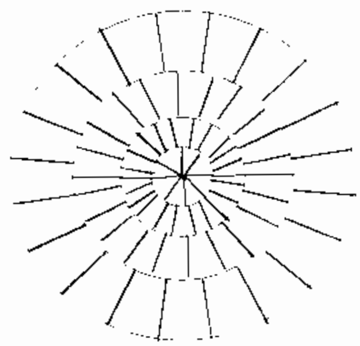


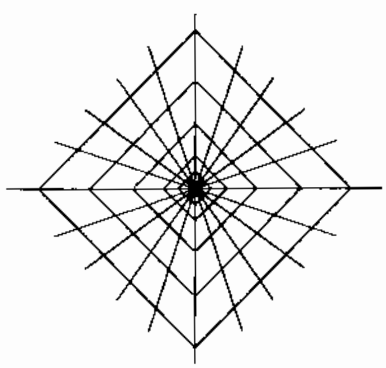


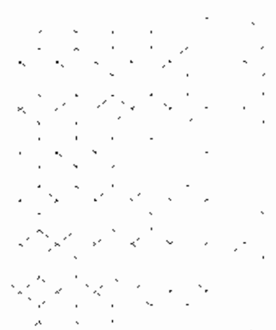


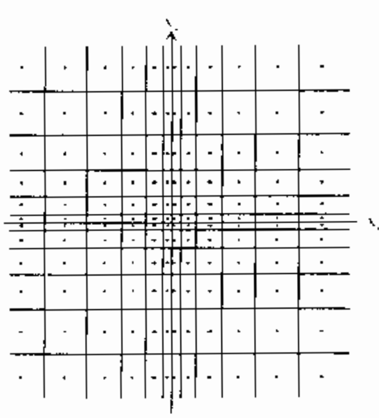


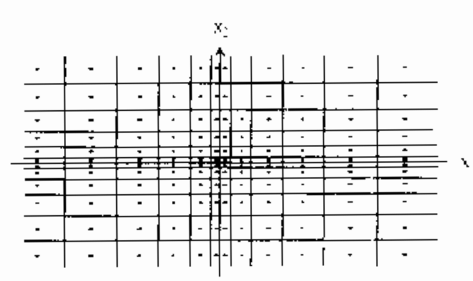


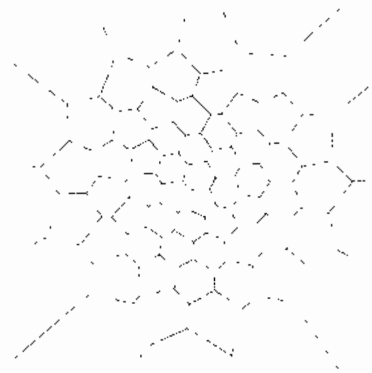


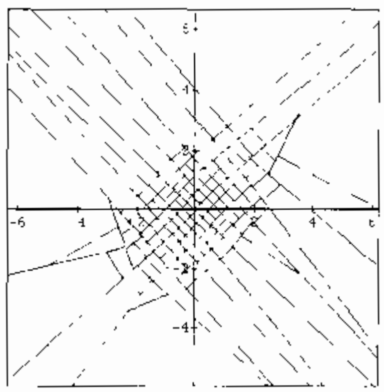


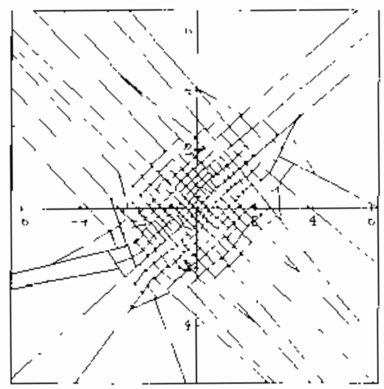








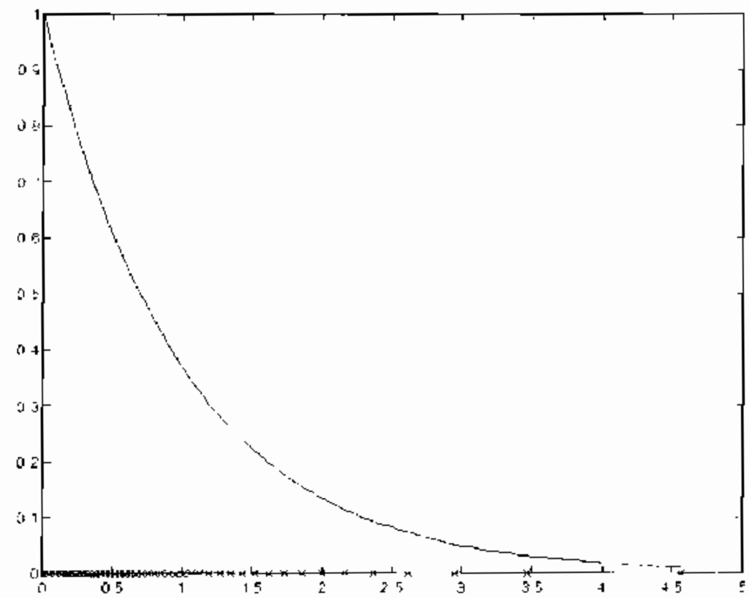




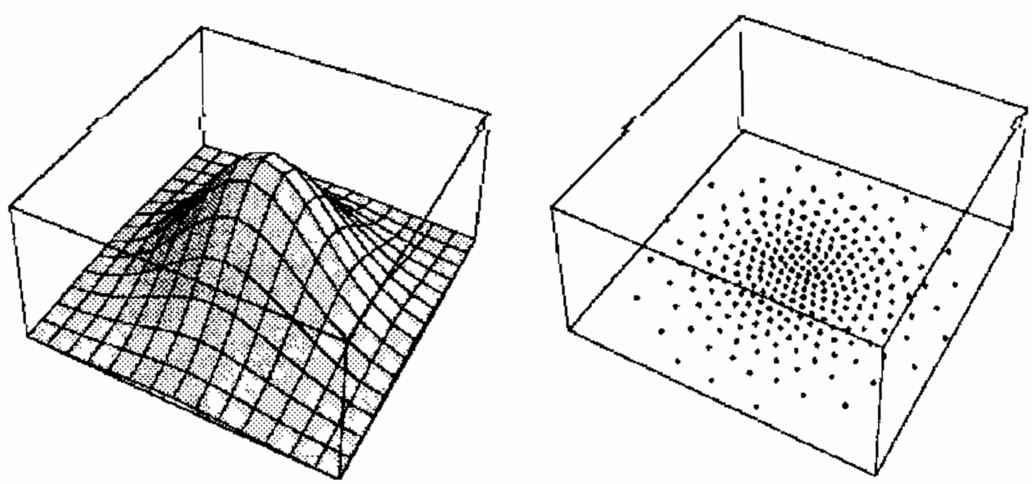
Scalar Quantizer (k=1)

Examples of Point Densities

k=1



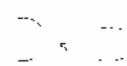
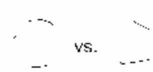
Optimal Quantizer for a 2-dimensional IID Gaussian random vector and its point density



- The NMI of various cell shapes

cell shape	dimension	NMI
1×2 rectangle	2	.104
cube	any	$.0833 = \frac{1}{12}$
hexagon	2	$.0802 = \frac{5\sqrt{3}}{108}$
circle	2	$.0796 = \frac{1}{4\pi}$
sphere	3	$.0770 = \frac{(4\pi/3)^{-2/3}}{5}$
sphere	k	$\frac{1}{(k+2)(V_k)^{2/k}}$
sphere	∞	$.0585 = \frac{1}{2\pi e} \simeq \frac{1}{17}$
$s_1 \times s_2 \times \ldots \times s_k$ rectangle	k	$\frac{1}{12} \frac{\frac{1}{k}\sum_{i=1}^{k}(s_i)^2}{\left(\prod_{i=1}^{k}(s_i)^2\right)^{1/k}} = \frac{1}{12} \frac{\text{arith mean of sides}^2}{\text{geom mean of sides}^2}$

where V_k = volume of k-dimensional sphere with radius 1

- Shapes that tend to make NMI smaller
  + Spheroidal rather than oblong 
  + More finely faceted (many sides rather than few) 

  + Higher rather than lower dimension
- A sphere has the lowest NMI of any cell of a given dimension.
- NMI of a sphere decreases with dimension to the limit $1/2\pi e = .0585$

The Best Known Tesselating Polytopes

dimension	polytope	m_K^*	best known, (upper bound)	conj'd lower bound	sphere lower bound	gain (dB) $10 \log m_i^* / m_K^*$
1	interval	0833'			.0833	0
2	hexagon	.0802'			.0796	16
3	unknown		.0785' truncated octahedron	077875	.0770	26
4	"		0766	0.0761'	.0750	39
5	"		.0756	0 0747'	.0735	47
6	"		.0742	0.0735'	.0723	.55
7	"		.0731	0 0725'	.0713	60
8	"		.0717	0.0716'	0705	66
12	"		.0701	0 0692'	0691	81
16	"		.0683	0.0676'	0666	.91
24	"		.0658	0 0656'	.0647	1.10
5					.0623'	1 26
1					0608'	1 37
2					0599'	1.43
3					.0595'	1 46
very large	sphere	0585'			.0585	1.53