April 26, 2005

CHI Mathematics of Information talk: "Analytic Combinatorics"

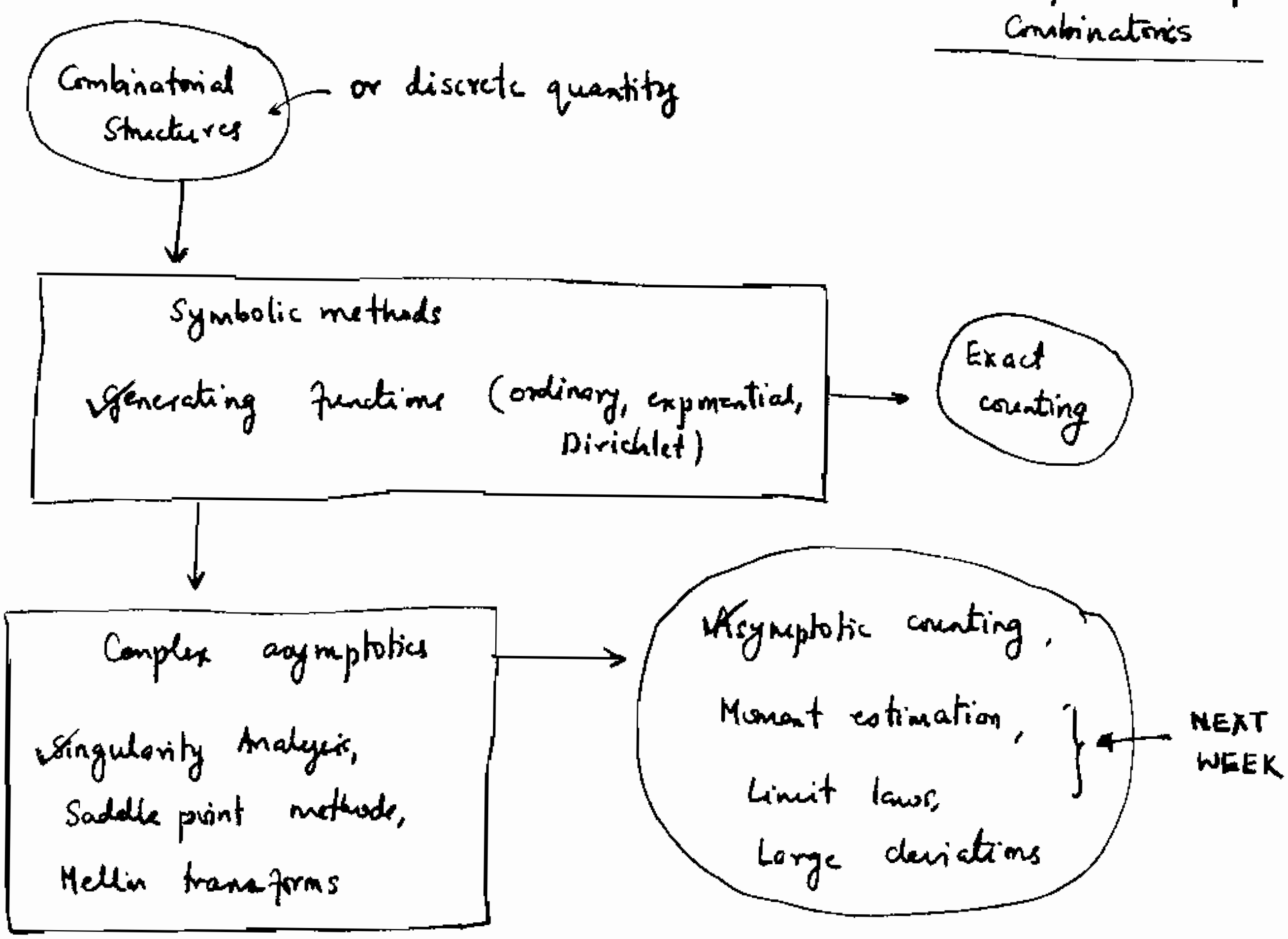
References: Analytic Combinatorics by Flajolet, Sedgewick

Singularity Analysis of generating functions by Flajolet, Odlyzko

SA and asymptotics of Bernoulli sums by Flajolet

SA, Hadamard products, and tree recurrences by FFK

Landscape of Analytic Combinatorics



f_n : - quantity of interest indexed by integer n
- interested in f_n as $n \to \infty$

$f(z)$: appropriate generating function :

OGF: $f(z) := \sum_n f_n z^n$

EGF: $f(z) := \sum_n f_n \frac{z^n}{n!}$ (labeled structures)

[SKIP] Dirichlet: $f(z) := \sum_n \frac{f_n}{n^z}$ (analytic number theory)

Step 1 : generating function

- using the description of f_n "obtain" the generating function $f(z)$ (as a formal power series)

Example 1

Fibonacci numbers : $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$

$f_0 = f_1 = 1$

OGF : $f(z) = \frac{1}{1-z-z^2}$

Example 0.5

f_n : # ways of arranging n pairs of parentheses ( ), into a legal string

- Given a legal string, scan from left to right and find the smallest k such that the first $2k$ chars. are themselves legal.

- Then
$$f_n = \sum_{k} f_{k-1}\, f_{n-k} \quad {}_{\wedge}^{n \geq 1} \qquad \text{with} \quad f_0 = 1$$

primitive legal strings of length $2k$
= # legal strings of length $2(k-1)$

OGF:

$$F(z) - 1 = z\, F^2(z)$$

$$\Rightarrow \qquad f(z) = \frac{1 - \sqrt{1-4z}}{2z} .$$

Example 1

f_n : # 2 regular $_{\wedge}^{\text{labeled, simple}}$ graphs on n vertices (labeled)

- Every 2-regular graph is a disjoint union of undirected cycles.

d_n : # $_{\wedge}^{\text{undirected}}$ cycles of length n.

$$d_n = 0 \quad \text{for} \quad \cancel{n \geq} \quad n \leq 2.$$

For $n \geq 3$, $\quad d_n = \underbrace{\frac{n!}{n}}_{\text{cyclic shift}} \Big/ \underbrace{2}_{\text{reversal}}$

EGF:

So
$$d(z) = \sum_{n \geq 3} d_n \frac{z^n}{n!} = \sum_{n \geq 3} \frac{(n-1)!}{2\,n!} z^n$$

$$= \frac{1}{2} \sum_{n \geq 3} \frac{z^n}{n} = \frac{1}{2} \left\{ \ln \frac{1}{1-z} - z - \frac{z^2}{2} \right\}$$

By the exponential formula, the EGF of f_n is given by exp

$$f(z) = \exp d(z) = \frac{e^{-z/2 - z^2/4}}{\sqrt{1-z}}.$$

2-regular graphs w/ exactly k cycles: $\dfrac{d(z)^k}{k!}$

Example 2 Entropy of the Binomial distribution

$$H_{n,p} = - \sum_{k=0}^{n} \pi_{n,k} \ln \pi_{n,k}, \quad \text{where} \quad \pi_{n,k} := \binom{n}{k} p^k q^{n-k}$$

$q = 1-p$

(fixed p, $n \to \infty$)

$$= - \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \left[\cancel{\ln \binom{n}{k}} + k \ln p + (n-k) \ln q \right]$$

$\ln n! - \ln k! - \ln (n-k)!$

$$= -\ln n! \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} - \ln p \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$- \ln q \sum_{k=0}^{n} (n-k) \binom{n}{k} p^k q^{n-k}$$

$$+ \sum_{k=0}^{n} \binom{n}{k} (\ln k!)\, p^k q^{n-k} + \sum_{k=0}^{n} \binom{n}{k} \ln (n-k)!\, p^k q^{n-k}$$

$$= -\ln n! - np \ln p - nq \ln q$$

$$+ S_{n,p}^{[\ln k!]} \cancel{\text{and}} + S_{n,q}^{[\ln k!]}, \qquad \leftarrow \boxed{\text{SAVE}}$$

where $$S_{n,p}^{[g_k]} := \sum_{k=0}^{n} \binom{n}{k} \overset{g_k}{\cancel{}}\, p^k \underset{(1-p)}{\cancel{q}}^{\,n-k}. \quad \Big\}\ \text{expected value of } g_{Bi(n,p)}$$

– To know the asymptotics of $\overset{H_{n,p}}{\cancel{}}$ we need those of $S_{n,p}$.

Define gfs: $$S(z) := \sum_n S_{n,p}[g_k]\, z^n$$

and $$g(z) := \sum_k g_k z^k$$

Easy to check:

$$S(z) = \frac{1}{1-qz}\, g\left(\frac{pz}{1-qz}\right).$$

$$\text{RHS} = \sum_k g_k\, p^k (1-qz)^{-k-1} z^k$$

$$= \sum_k g_k\, p^k z^k \sum_n \binom{n}{k} q^{n-k} z^{n-k}$$

$$= \sum_n z^n \underbrace{\sum_k \binom{n}{k} g_k\, p^k q^{n-k}}_{S_n} = \text{RHS}$$

What's $g(z)$?

$$g(z) = \sum_k (\log k!)\, z^k = \sum_k \left(\sum_{j=1}^{k} \log j \right) z^k$$

Define $L(z) := \sum_{n \geq 1} (\ln n)\, z^n$

Then $g(z) = \dfrac{1}{1-z} L(z)$.

(Example 3 if time permits) $\boxed{\text{No!}}$

$$\cancel{(1-qz)^{-k-1} = \sum_n \binom{-k-1}{n} (-1)^{n-k} q^{n-k} z^{n-k}}$$

$$\cancel{(-1)^{n-k} \frac{(-k-1)(-k-2)\cdots(-k-n)}{n!}} = \cancel{(-1)^k \frac{(k+1)(k+2)\cdots(k+n)}{n!}}$$

SUMMARY OF EXAMPLES

0. $$f(z) = \frac{1}{1-z-z^2}$$

0.5 $$f(z) = \frac{1-\sqrt{1-4z}}{2z}$$

1 $$f(z) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}} \qquad \text{(EGF)}$$

2 $$S(z) = \frac{1}{1-qz}\, g\left(\frac{pz}{1-qz}\right), \quad \text{where}$$

$$g(z) = \frac{1}{1-z} L(z), \quad \text{where} \quad L(z) := \sum_k (\ln k)\, z^k.$$

- So far, our gfs have been <u>formal</u> objects.
- Assigning complex values to the variable in the gf has good consequences:

  the behavior of the function near its singularities provides information about the function's coefficient

<u>Notation</u>: $[z^n]\, f(z) :=$ coefficient of z^n in $f(z)$.

Basic singularity analysis :

Given $f(z)$, • analytic at the origin
• radius of convergence 1
• + technical condition

- Expand the function $f(z)$ around its singularity :

$$f(z) = \sigma(z) + O(|\tau(z)|)$$

where $\sigma(z) \gg \tau(z)$ as $z \to 1$ and both

$\sigma(z)$ and $\tau(z)$ are "simple functions"

* (Our simple functions will be of the form $(1-z)^{+\alpha} \left[\ln (1-z)^{-1} \right]^{\beta}$.)

- Take formal Taylor coefficients on both sides

$$f_n = [z^n]\, f(z) = [z^n]\, \sigma(z) + [z^n]\, O(|\tau(z)|)$$

To know the asymptotics of f_n, we need two things:

1. A catalog of exact and/or asymptotic forms of standard functions;

2. A way of extracting asymptotics of coefficient of functions known only by their order of growth, near those singularities

Under favorable conditions we will be able to say

$$f(z) = \sigma(z) + O(|\tau(z)|) \Rightarrow f_n = \sigma_n + O(\tau_n),$$

where $\sigma_n = [z^n]\,\sigma(z)$ and $\tau_n = [z^n]\,\tau(z)$.

$\boxed{\text{SAVE}}$

Asymptotic forms of standard functions:

- $$[z^n]\,(1-z)^{\alpha} = \binom{n-\alpha-1}{-\alpha-1} \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left[1 + \frac{\alpha(\alpha+1)}{2n} + \frac{\alpha(\alpha+1)(\alpha+2)(3\alpha+1)}{24n^2} + \cdots\right]$$

  if $\alpha \notin \{0, 1, 2, \ldots\}$

- Note that, ~~formally~~

pos. integer

$$(1-z)^{\alpha}\left(\ln\frac{1}{1-z}\right)^{k} = (-1)^{k}\frac{\partial}{\partial\alpha^{k}}(1-z)^{\alpha}$$

$$= (-1)^{k}\frac{\partial}{\partial\alpha^{k}}\binom{n-\alpha-1}{-\alpha-1}$$

$$= (-1)^{k}\frac{\partial}{\partial\alpha^{k}}\frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\,\Gamma(n+1)}$$

Formally

$$[z^n]\,(1-z)^{\alpha}\left(\ln\frac{1}{1-z}\right)^{k} = (-1)^{k}\frac{\partial}{\partial\alpha^{k}}$$

$$\Gamma(s) := \int_0^\infty e^{-t}\, t^{s-1}\, dt$$

$$\mathrm{Re}(s) > 0$$

$\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(s+1) = s\Gamma(s)$.

For r, a nonnegative integer:

$$(1-z)^\alpha \left(\ln \frac{1}{1-z}\right)^r = (-1)^r \frac{\partial}{\partial \alpha^r} (1-z)^\alpha$$

So, formally

$$[z^n]\ (1-z)^\alpha \left(\ln \frac{1}{1-z}\right)^r = (-1)^r\ [z^n] \frac{\partial}{\partial \alpha^r} (1-z)^\alpha$$

needs justification

$$= (-1)^r \frac{\partial}{\partial \alpha^r} [z^n]\ (1-z)^\alpha$$

$$= (-1)^r \frac{\partial}{\partial \alpha^r} \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\,\Gamma(n+1)}.$$

Transfer theorems:

Sufficient condition for

$$f(z) = O\left(|1-z|^\alpha \left[\log (1-z)^{-1}\right]^\beta \right)$$

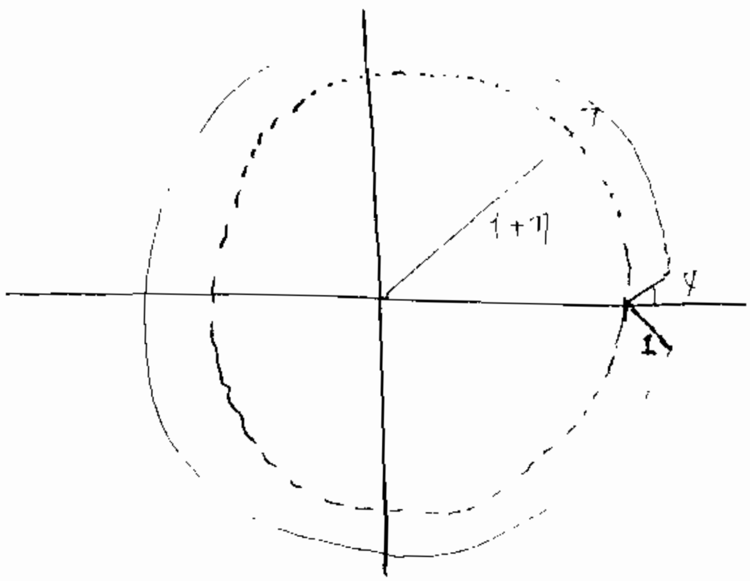
$$\Rightarrow \quad f_n = O\left(n^{-\alpha-1} (\log n)^\beta \right).$$

Δ- regularity

<u>Definition</u> A function defined by a Taylor series with radius of convergence 1 is <u>Δ-regular</u> if it can be analytically continued in a domain

$$\Delta(\phi, \eta) := \{ z : |z| < 1 + \eta, |\operatorname{Arg}(z-1)| > \phi \}$$

for some $\eta > 0$ and $0 < \phi < \frac{\pi}{2}$.



" Pacman region"
" Camembert region"
" indendented crown"

<u>Theorem</u>. (Flajolet, Odlyzko, 1990)

If f is Δ-regular and as $z \to 1$ in $\Delta(\phi, \eta)$

$$f(z) = O\left(|1-z|^{\alpha} \left|\log (1-z)^{-1}\right|^{\beta} \right)$$

Then $f_n = [z^n] f(z) = O\left(n^{-\alpha-1} (\log n)^{\beta} \right)$ as $n \to \infty$.

[Little-o also works].

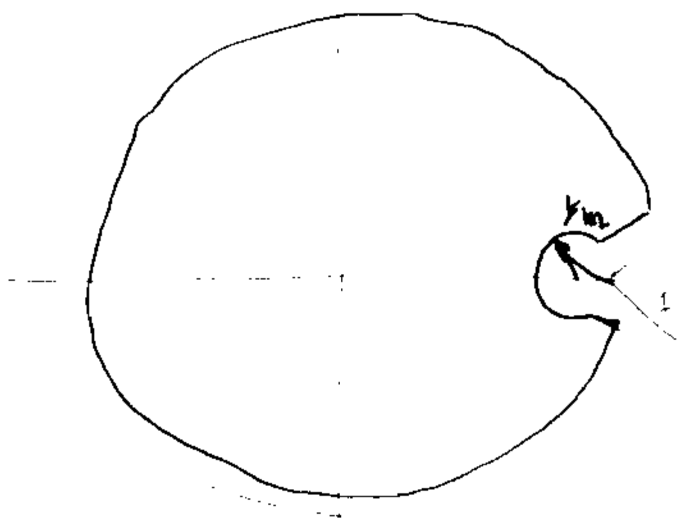
Proof idea

Cauchy integral formula:

$$f_n = [z^n]\, f(z) = \frac{1}{2\pi i} \int_\gamma f(z) \frac{dz}{z^{n+1}}$$

any closed curve inside $\Delta(\phi, \eta)$.

γ:



dominant contributions from the rectilinear part and small circle.

$\square$

Putting everything together: $\alpha \notin \{0, 1, 2, \dots\}$

As $z \to 1$ in Δ-domain

$$f(z) = (1-z)^\alpha \Rightarrow f_n = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}$$

$$f(z) = O(|1-z|^\alpha) \Rightarrow f_n = O(n^{-\alpha-1})$$

$$f(z) = o(|1-z|^\alpha) \Rightarrow f_n = o(n^{-\alpha-1})$$

$$f(z) \sim (1-z)^\alpha \Rightarrow f_n \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}.$$

as $n \to \infty$

<u>Fact.</u> Many gfs ~~are~~ that are compositions of

$$\underbrace{\frac{1}{1-z}}_{\mathbb{C}\setminus\{1\}}, \quad \underbrace{\exp(z)}_{\text{entire}}, \quad \underbrace{\ln\frac{1}{1-z}, \quad \sqrt{1-z}}_{\mathbb{C}\setminus[1,\infty]} \quad \text{are}$$

Δ- regular.

(OK as long as growth near the singularity is only polynomial)
$\exp\left(\frac{z}{1-z}\right)$ is excluded; covered by saddlepoint

<u>Examples</u>

0. $$f(z) = \frac{1}{1-z-z^2} = -\frac{1}{(\phi_1-z)(\phi_2-z)} \qquad \text{with } |\phi_1|<|\phi_2|$$

Singularities: ϕ_1, ϕ_2 (poles)

Dominant singularity: ϕ_1. (closest to the origin)

<u>Scale</u>: $$g(z) := f(\phi_1 z) = -\frac{1}{(\phi_1 - z\phi_1)(\phi_2 - z\phi_1)}$$

Now the dominant singularity is at $z=1$. g is Δ-regular.

(can be analytically continued to $\mathbb{C}\setminus\{1, \frac{\phi_2}{\phi_1}\}$)

$$g(z) \sim -\frac{1}{\phi_2-\phi_1}\,\frac{1}{\phi_1}\,(1-z)^{-1}$$

$\Rightarrow$ By SA $\quad g_n \sim \dfrac{1}{\phi_1(\phi_1-\phi_2)}$

$\dfrac{1}{\text{golden ratio}}$

and $\quad g_n = \phi_1^n f_n$

so $\quad f_n \sim \dfrac{1}{\phi_1(\phi_1-\phi_2)}\,\phi_1^{-n}.$

Example 0.5 $\qquad f(z) = \dfrac{1-\sqrt{1-4z}}{2z} \qquad f_n = \dfrac{1}{n+1}\binom{2n}{n}$

$$g(z) := f\left(\frac{z}{4}\right) = 2\,\frac{1-\sqrt{1-z}}{z}$$

Singularity: 1 (dominant, branch type)

Δ-regular: can be analytically continued to $\mathbb{C} \setminus [1,\infty]$

Singular expansion:

$$z = 1-(1-z)$$

$$g(z) = 2\left[1-(1-z)^{1/2}\right]\left\{1+(1-z)+O(|1-z|^2)\right\}$$

$$= 2\left[1 \cancel{\bullet} - (1-z)^{1/2} + (1-z) + O(|1-z|^{3/2})\right]$$

By SA

$$g_n = -2\,\frac{n^{-3/2}}{\Gamma(-\frac{1}{2})} + O\left(n^{-5/2}\right) \qquad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$$

$$= \frac{n^{-3/2}}{\sqrt{\pi}} + O(n^{-5/2})$$

$$f_n = \frac{4^n\, n^{-3/2}}{\sqrt{\pi}} + O(4^n\, n^{-5/2}) \quad . \quad \checkmark$$

$\underline{1}$ $$\frac{e^{-z/2-z^2/4}}{\sqrt{1-z}} = f(z)$$ (EGF)

- $e^{-z/2-z^2/4}$ is entire
- $(1-z)^{-1/2}$ is Δ-regular (analytic in the complex plane slit along $[1,\infty]$)

$\Rightarrow$ $f(z)$ is Δ-regular with its dominant singularity at $z=1$.

Singular expansion: As $z \to 1$

$$e^{-z/2-z^2/4} = e^{-3/4} + e^{-3/4}(1-z) + O(|1-z|^2)$$

$$\Rightarrow f(z) = e^{-3/4}(1-z)^{-1/2} + e^{-3/4}(1-z)^{1/2} + O(|1-z|^{3/2})$$

$$\Rightarrow [z^n]\, f(z) = \frac{e^{-3/4}}{\sqrt{\pi n}}\left[1 - \frac{1}{8n} + \frac{1}{128n^2} + \cdots\right]$$

$$- \frac{e^{-3/4}}{2\sqrt{\pi n^3}}\left[1 + \frac{3}{8n} + \cdots\right]$$ and

$f_n = n!\,(\quad)$ ✓

$$+ O(n^{-5/2})$$

$$= \frac{e^{-3/4}}{\sqrt{\pi n}} - \frac{5\,e^{-3/4}}{8\sqrt{\pi n^3}} + O(n^{-5/2})$$

□

2. $$S(z) = \frac{1}{1-qz}\, g\left(\frac{pz}{1-qz}\right), \text{ where}$$

$$g(z) = \frac{1}{1-z} L(z), \quad \text{where} \quad L(z) = \sum_k (\ln k)\, z^k.$$

- $z \mapsto \dfrac{pz}{1-qz}$ is a conformal map mapping the unit disk to the disk with diameter $\left[-\frac{p}{1+q}, 1\right]$.

- If $g(z)$ has an isolated singularity at $z=1$ and is Δ-regular then ~~so is~~ the same is true of $S(z)$.

SUMMARY: If $g(z)$ is 'amenable' then so is $S(z)$.

What about $g(z)$? For that we need to look at $L(z)$.

Definition.
$$Li_{\alpha,r}(z) := \sum_{n \geq 1} \frac{(\ln n)^r}{n^\alpha} z^n$$

generalized polylogarithm

r: is a nonnegative integer

α: arbitrary complex number

$\{$ Note that $L(z) = Li_{0,1}(z)$. $\}$

Theorem. $\mathrm{Li}_{\alpha,r}(z)$ is analytic in the split plane $\mathbb{C} \setminus [1,+\infty)$

For $\alpha \neq 1, 2, \ldots$ its singular expansion is

$$\mathrm{Li}_{\alpha,0}(z) \sim \Gamma(1-\alpha)\, t^{\alpha-1} + \sum_{j\geq 0} \frac{(-1)^j}{j!} \zeta(\alpha-j)\, t^j \qquad (*)$$

$$t := -\ln z = \sum_{\ell=1}^{\infty} \frac{(1-z)^\ell}{\ell}$$

For $r>0$, the singular expansion of $\mathrm{Li}_{\alpha,r}(z)$ is obtained by

$$\mathrm{Li}_{\alpha,r}(z) = (-1)^r \frac{\partial^r}{\partial \alpha^r} \mathrm{Li}_{\alpha,0}(z),$$

termwise diff of $(*)$.

Corollary. $L(z)$ is analytic in the split plane

$\mathbb{C}\setminus[1,\infty)$ and as $z \to 1$, ...

[illegible]

[illegible]

[illegible]

- get an expansion for $L(z)$ [illegible]
- then, one for $g(z) = (1-z)^{-1} L(z)$
- then one for $S(z) = \frac{1}{1-qz}\, g\left(\frac{pz}{1-qz}\right)$

After some calculation, we find

$$\cancel{S(z) = p(1-z)^{-2}\left[\ln(1-z)^{-1} + \ln p + \gamma\right] + (1-z)^{-1}(p-\tfrac{1}{2})}$$

$$\begin{aligned} S(z) = {} & p \cdot (1-z)^{-2} \ln (1-z)^{-1} \\ & + p(\ln p - \gamma)(1-z)^{-2} \\ & + (p - \tfrac{1}{2})(1-z)^{-1} \ln(1-z) \\ & + \left[(\tfrac{1}{2} - p)(1-\gamma) - p \ln p + \ln \sqrt{2\pi p} \right] (1-z)^{-1} \\ & + O(|\log (1-z)^{-1}|). \end{aligned}$$

$\boxed{\gamma \text{ shows up because } \Gamma'(1) = -\gamma}$

Thus, by SA

$$S_{n,p}[\ln k!] = \ln \left[(pn)^{pn} e^{-pn} \sqrt{2\pi pn} \right] - \frac{p-1}{2} + O\left(\frac{\log n}{n}\right)$$

$S_{n,q}[\ln k!]$ follows immediately and using Stirling:

$$H_{n,p} = \frac{1}{2} \ln n + \frac{1}{2} + \ln \sqrt{2\pi p(1-p)} + O\left(\frac{\log n}{n}\right)$$

Other applications:

- mean, variance estimates of functions of binomial random variables: $\frac{1}{X}$, $\ln X$, $\sqrt{X}$ ...

<u>SUMMARY</u>.

- Singularities of gf $\leftrightarrow$ asymptotics of underlying sequence

<u>Other closure properties</u>:

- Integration, differentiation
- ★ Hadamard Products.

- Functions that get large: Saddle points $\square$
- Harmonic sums: Mellin transforms

<u>NEXT WEEK</u> Limit laws via the Method of Moments